

# Long-time asymptotics for nonlinear growth-fragmentation equations

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January 12, 2013

## Abstract

We are interested in the long-time asymptotic behavior of growth-fragmentation equations with a nonlinear growth term. We present examples for which we can prove either the convergence to a steady state or conversely the existence of periodic solutions. Thanks the General Relative Entropy method applied to well chosen self-similar solutions, we show that the equation can “asymptotically” be reduced to a system of ODEs. Then stability results are proved by using a Lyapunov functional, and existence of periodic solutions are proved thanks to the Poincaré-Bendixon theorem or by Hopf bifurcation.

**Keywords:** size-structured populations, growth-fragmentation processes, eigenproblem, self-similarity, relative entropy, long-time asymptotics, stability, periodic solution, Poincaré-Bendixon theorem, Hopf bifurcation.

**AMS Class. No.** 35B10, 35B32, 35B35, 35B40, 35B42, 35Q92, 37G15, 45K05, 92D25

## 1 Introduction

We are interested in growth models which take the form of a *mass preserving* fragmentation equation complemented with a transport term. Such models are used to describe the evolution of a population in which each individual grows and splits or divides. The individuals can be for instance cells [4, 5, 34, 45] or polymers [7, 17, 31] and are structured by a variable  $x > 0$  which may be size [21, 22], label [2], protein content [19, 42], proliferating parasites content [3] ; etc. More precisely, we denote by  $u = u(t, x) \geq 0$  the density of individuals of structured variable  $x$  at time  $t$ , and we consider that the time dynamic of the population is given by the following equation

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) + \frac{\partial}{\partial x} (\tau(t, x) u(t, x)) + \mu(t, x) u(t, x) = (\mathcal{F}u)(t, x), & t \geq 0, x > 0, \\ u(t, x = 0) = 0, & t \geq 0, \\ u(t = 0, x) = u_0(x) \geq 0, & x > 0, \end{cases} \quad (1)$$

where  $\mathcal{F}$  is a *mass conservative* fragmentation operator

$$(\mathcal{F}u)(t, x) = \int_x^\infty b(t, y, x) u(t, y) dy - \beta(t, x) u(t, x). \quad (2)$$

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The *mass conservation* for the fragmentation operator requires the relation

$$\beta(t, x) = \int_0^x \frac{y}{x} b(t, x, y) dy. \quad (3)$$

The coefficient  $\beta(t, y) \geq 0$  represents the rate of splitting for a particle of size  $y$  at time  $t$  and  $b(t, y, x) \geq 0$  represents the formation rate of a particle of size  $x \leq y$  after the fragmentation. The velocity  $\tau(t, x) > 0$  in the transport term represents the growth rate of each individual, and  $\mu(t, x) \geq 0$  is a degradation or death term.

We consider that the time dependency of  $\tau$  and  $\mu$  is of the form

$$\tau(t, x) = V(t)\tau(x) \quad \text{and} \quad \mu(t, x) = R(t)\mu(x) \quad (4)$$

and moreover that the size dependency is a powerlaw

$$\tau(x) = \tau x^\nu \quad \text{and} \quad \mu(x) \equiv \mu. \quad (5)$$

The choice of the coefficients  $V(t)$  and  $R(t)$  depends on the cases we want to analyse. We give below four examples in which they are nonlinear terms or periodic controls. The fragmentation coefficients are assumed to be time-independent and of *self-similar* form

$$b(t, x, y) = \frac{\beta(x)}{x} \kappa\left(\frac{y}{x}\right) \quad (6)$$

with  $\kappa$  a nonnegative measure on  $[0, 1]$ . For  $b$  under this form, the quantity

$$n_0 := \int_0^1 \kappa(z) dz$$

represents the mean number of fragments produced by the fragmentation of an individual. We remark that under Assumption (6), relation (3) becomes  $\int_0^1 z \kappa(z) dz = 1$  and then it enforces  $n_0 > 1$ . If  $\kappa$  is symmetric ( $\kappa(z) = \kappa(1 - z)$ ), we have necessarily  $n_0 = 2$ . We also assume that  $\beta$  is a powerlaw coefficient

$$\beta(x) = \beta x^\gamma \quad (7)$$

and we denote by  $\mathcal{F}_\gamma$  the fragmentation operator associated to coefficients satisfying Assumptions (6) and (7).

Now we state our main results concerning different choices for  $V(t)$  and  $R(t)$ . First we investigate the nonlinear growth-fragmentation equations corresponding to the case when  $V$  or/and  $R$  are functions of the solution  $u(t, x)$  itself. We also consider a model of *polymerization* in which the transport term depends on  $u$  and on a solution to an ODE coupled to the growth-fragmentation equation. The long-time asymptotic behavior of these equations is investigated under the assumption that  $\tau(x)$  is linear (*i.e.*  $\nu = 1$ ) and  $\beta$  increasing (*i.e.*  $\gamma > 0$ ). We finish with a study of the long-time asymptotics in the case when  $V$  and  $R$  are known periodic controls.

**Example 1. Nonlinear drift-term.** We consider that the death rate is time independent ( $R \equiv 1$ ) and that the transport term depends on the solution itself through the nonlinearity

$$\frac{\partial}{\partial t}u(t, x) = -f\left(\int x^p u(t, x) dx\right) \frac{\partial}{\partial x}(x u(t, x)) - \mu u(t, x) + \mathcal{F}_\gamma u(t, x) \quad (8)$$

where  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a continuous function which represents the influence of the wheighted total population  $\int x^p u(t, x) dx$  ( $p \geq 0$ ) on the growth process. Such weak nonlinearities are common in structured populations (see [36, 37, 60] for instance). The stability of the steady states for related models has already been investigated (see [18, 26, 27, 28, 50]) but never for the growth-fragmentation equation with the nonlinearities considered here. We prove in Section 3 convergence and nonlinear stability results for Equation (8) in the functional space  $\mathcal{H} := L^2((x + x^r)dx)$  for  $r$  large enough, or more precisely in its positive cone denoted by  $\mathcal{H}^+$ . These results are stated in the two following theorems. They require that  $f$  is continuous and satisfies

$$\mathcal{N} := \{I; f(I) = \mu\} \text{ is a finite set} \quad \text{and} \quad \limsup_{I \rightarrow \infty} f(I) < \mu. \quad (9)$$

**Theorem 1.1** (Convergence). *Assume that  $f$  satisfies Assumption (9), that  $\gamma \in (0, 2]$  and that the fragmentation kernel  $\kappa$  satisfies Assumption (32). Then the number of positive steady states of Equation (8) is equal to  $\sharp \mathcal{N}$  and any solution with an initial distribution  $u_0 \in \mathcal{H}^+$  either converges to one of these steady states or vanishes.*

**Theorem 1.2** (Local stability). *Assume that  $f \in \mathcal{C}^1(\mathbb{R}_+)$  satisfies Assumption (9), that  $\gamma \in (0, 2]$  and that the fragmentation kernel  $\kappa$  satisfies Assumption (32). Then the trivial steady state is locally exponentially stable if  $f(0) < \mu$  and  $p \geq 1$ , and unstable if  $f(0) > \mu$ . Any nontrivial steady state  $u_\infty$  is locally asymptotically stable if  $f'(\int x^p u_\infty(x) dx) < 0$ , locally exponentially stable if additionally  $\kappa \equiv 2$ , and unstable if  $f'(\int x^p u_\infty(x) dx) > 0$ .*

As an immediate consequence of these two theorems, we have the following corollary.

**Corollary 1.3** (Global stability). *For  $\gamma \in (0, 2]$  and under Assumption (32), if  $f \in \mathcal{C}^1(\mathbb{R}_+)$  satisfies (9) with  $\mathcal{N}$  a singleton, then there is a unique nontrivial steady state  $u_\infty$ , and if additionally  $f'(\int x^p u_\infty(x) dx) < 0$ , then it is globally asymptotically stable in  $\mathcal{H}^+ \setminus \{0\}$ .*

The method combines several arguments. First it uses the General Relative Entropy principle introduced by [48, 49, 51] for the linear case. Secondly it reduces the system to a set of ODEs which has the same asymptotic behavior than Equation (8). Then we build a Lyapunov functional for this reduced system. Therefore our result extends to the case of the growth-fragmentation equation several satability results proved for the nonlinear renewal equation in [47, 53].

**Example 2. Nonlinear drift and death terms.** We can also treat several nonlinearities as in

$$\frac{\partial}{\partial t}u(t, x) = -f\left(\int x^p u(t, x) dx\right) \frac{\partial}{\partial x}(x u(t, x)) - g\left(\int x^q u(t, x) dx\right) u(t, x) + \mathcal{F}_\gamma u(t, x). \quad (10)$$

In this case, we show that oscillating behaviors can appear. The existence of non-trivial periodic solution for structured population models is a very interesting and difficult problem. It has been mainly investigated for age structured models with nonlinear renewal and/or death terms, but there are very few results [1, 6, 16, 38, 43, 44, 54, 57, 58]. Taking advantage of the Poincaré-Bendixon theorem, we prove the existence of oscillating solutions for Equation (10) and the “stability” of this behavior in a sense specified in the following theorem.

**Theorem 1.4.** *For  $\gamma \in (0, 2]$  and under Assumption (32), there exist functions  $f$  and  $g$  and parameters  $p$  and  $q$  for which we can find an open set  $\mathcal{V} \subset \mathcal{H}^+$  with the property that any solution to Equation (10) with an initial distribution  $u_0 \in \mathcal{V}$  converges to a periodic solution.*

The proof of this result is given in Section 4.

**Example 3. The prion equation.** In Section 5, we are interested in a general so-called prion equation

$$\begin{cases} \frac{dV(t)}{dt} = -V(t)f\left(\int x^p u\right) \int_0^\infty xu(t, x) dx - \delta V(t) + \lambda, \\ \frac{\partial}{\partial t}u(t, x) = -V(t)f\left(\int x^p u\right) \frac{\partial}{\partial x}(xu(t, x)) - \mu u(t, x) + \mathcal{F}_\gamma u(t, x). \end{cases} \quad (11)$$

In this equation, the growth term depend on the quantity  $V(t)$  of another population (monomers for the prion proliferation model). We prove for this system the existence of periodic solutions under some conditions on  $f$ . In age structured models, such solutions are usually built using bifurcation theory, particularly by Hopf bifurcation (see [41] for a general theorem). Here we consider the power  $p$  as a bifurcation parameter and we prove existence of periodic solution by Hopf bifurcation.

**Theorem 1.5.** *There exist a function  $f$  and parameters for which Equation (11) admits periodic solutions.*

**Example 4. Perron vs. Floquet.** Our method in Section 2.2 can also be applied to the situation when  $V(t)$  and  $R(t)$  are periodic controls

$$\frac{\partial}{\partial t}u(t, x) + V(t)\frac{\partial}{\partial x}(\tau x u(t, x)) + R(t)\mu u(t, x) = \mathcal{F}_\gamma u(t, x). \quad (12)$$

In this case, Theorem 2.1 allows to build a particular solution called the Floquet eigenvector, starting from the Perron eigenvector which correspond to constant parameters. Moreover, we can compare the associated Floquet eigenvalue to the Perron eigenvalue. The results about this problem are stated in Section 6.

Before treating the different examples, we explain in Section 2 the general method used to tackle these problems. It is based on the main result in Theorem 2.1 and the use of the eigenelements of the growth-fragmentation operator together with General Relative Entropy techniques. In Sections 3, 4, 5 and 6, we give the proofs of the results in Examples 1, 2, 3 and 4 respectively.

## 2 Technical Tools and General Method

### 2.1 Main Theorem

The proofs of the main theorems of this paper are based on the following result which requires to consider that  $\tau(x)$  is linear (*i.e.*  $\nu = 1$ ). In this case, there exists a relation between a solution to Equation (1) with time-dependent parameters  $(V_1(t), R_1(t))$  and a solution to the same equation with parameters  $(V_2(t), R_2(t))$ . More precisely, we can obtain one from the other by an appropriate dilation. The following theorem generalizes the change of variable used in [25] to build self-similar solution to the pure fragmentation equation.

**Theorem 2.1.** Consider that Assumptions (4)-(7) are satisfied with  $\nu = 1$  and  $\gamma > 0$ . For  $u_1(t, x)$  a solution to Equation (1) with parameters  $(V_1, R_1)$ , the function  $u_2(t, x)$  defined by

$$u_2(t, x) = W^{-k}(t)u_1\left(h(t), W^{-k}(t)x\right)e^{\mu \int_0^t (W(s)R_1(s) - R_2(s)) ds} \quad (13)$$

with  $k = \frac{1}{\gamma}$ , is a solution to Equation (1) with  $(V_2, R_2)$  if  $W > 0$  and  $h$  satisfy

$$\begin{aligned} \dot{W} &= \frac{\tau W}{k}(V_2 - V_1 W), \\ \dot{h} &= W. \end{aligned} \quad (14)$$

Conversely, if  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is one to one and if  $u_2$  is a solution with  $(V_2, R_2)$ , then  $u_1$  defined by (13) is a solution with  $(V_1, R_1)$ .

The proof of this result is nothing but easy calculation that we leave to the reader.

**Remark 2.2.** To check that  $h$  is one to one, we can take advantage that ODE (14) is a Bernoulli equation which can be integrated in

$$W(t) = \frac{W_0 e^{\frac{\tau}{k} \int_0^t V_2(s) ds}}{1 + W_0 \frac{\tau}{k} \int_0^t V_1(s) e^{\frac{\tau}{k} \int_0^s V_2(s') ds'} ds}. \quad (15)$$

To tackle the different examples, we use Theorem 2.1 together with two techniques appropriate for this type of equations. First we recall the existence of particular solutions to the growth-fragmentation equation in the case of time-independent coefficients. They correspond to eigenvectors to the growth-fragmentation operator and we can give their self-similar dependency on parameters in the case of powerlaw coefficients (see [30, 9]). This dependency is the starting point which leads to Theorem 2.1 and it allows to build an invariant manifold for Equation (1) in the case  $\nu = 1$ . It also provides interesting properties on the moments of the solutions when  $\nu \neq 1$ . Then we recall results about the General Relative Entropy (GRE) introduced by [48, 49, 51] for the growth-fragmentation model. This method ensures that the particular solutions built from eigenvectors are attractive for suitable norms.

## 2.2 Eigenvectors and Self-similarity: Existence of an Invariant Manifold

When the coefficients of Equation (1) do not depend on time, one can build solutions  $(t, x) \mapsto \mathcal{U}(x)e^{\Lambda t}$  by solving the Perron eigenvalue problem

$$\begin{cases} \Lambda \mathcal{U}(x) = -\frac{\partial}{\partial x}(\tau(x)\mathcal{U}(x)) - \mu(x)\mathcal{U}(x) + (\mathcal{F}\mathcal{U})(x), & x \geq 0, \\ \tau\mathcal{U}(x=0) = 0, & \mathcal{U}(x) \geq 0, & \int_0^\infty \mathcal{U}(x)dx = 1. \end{cases} \quad (16)$$

The existence of such elements  $\Lambda$  and  $\mathcal{U}$  has been first studied by [46, 52] and is proved for general coefficients in [23]. The dependency of these elements on parameters is of interest to investigate the existence of steady states for nonlinear problems (see [12, 9]). In the case of powerlaw coefficients, we can work out this dependency on frozen transport parameter  $V$  and death parameter  $R$  (see [30]). Under Assumptions (5)-(7), the necessary condition which appears in [23, 46] to ensure the existence of eigenelements is  $\gamma + 1 - \nu > 0$ . Then we define a *dilation parameter*

$$k := \frac{1}{\gamma + 1 - \nu} > 0 \quad (17)$$

and we have explicit self-similar dependencies

$$\Lambda(V, R) = V^{k\gamma} \Lambda(1, 0) - R\mu \quad \text{and} \quad \mathcal{U}(V; x) = V^{-k} \mathcal{U}(1; V^{-k}x). \quad (18)$$

The eigenvector  $\mathcal{U}$  does not depend on  $R$ , that is why we do not label it. Thereafter  $\Lambda(1, 0)$  and  $\mathcal{U}(1; \cdot)$  are denoted by  $\Lambda$  and  $\mathcal{U}$  for the sake of clarity. The result of Theorem 2.1 is based on the idea to use these dependencies to tackle time-dependent parameters. An intermediate result between the formula (13) and the dependencies (18) is given by the following corollary.

**Corollary 2.3.** *Under the assumptions of Theorem 2.1, if  $W$  is a solution to*

$$\dot{W} = \frac{\Lambda(W, 0)}{k} (V - W) \quad (19)$$

then

$$u(t, x) = \mathcal{U}(W(t); x) e^{\int_0^t \Lambda(W(s), R(s)) ds} \quad (20)$$

is a solution to Equation (1).

*Proof.* For  $\nu = 1$ , we can compute  $\Lambda = \Lambda(1, 0)$  by integrating Equation (16) with  $\mu \equiv 0$  against  $x dx$ . We obtain thanks to the *mass preservation* of  $\mathcal{F}$

$$\Lambda \int_0^\infty x \mathcal{U}(x) dx = \tau \int_0^\infty x \mathcal{U}(x) dx$$

and so  $\Lambda(1, 0) = \tau$ . Thus using the dependency (18) we find that  $\Lambda(W, 0) = \tau W$  and Equation (19) is nothing but a rewriting of Equation (14). We use this formulation (19) here to highlight the link with eigenlements, and because it allows after to obtain results in the cases when  $\nu \neq 1$ . Now we apply Theorem 2.1 for  $V_1 \equiv 1$ ,  $R_1 \equiv 0$  and  $V_2 \equiv V$ ,  $R_2 \equiv R$  and we obtain that

$$u_2(t, x) = W^{-k} \mathcal{U}(W^{-k}x) e^{\Lambda h(t) - \int_0^t R(s) ds} = \mathcal{U}(W(t); x) e^{\int_0^t \Lambda(W(s), R(s)) ds}$$

is a solution to Equation (1). □

This corollary provides a very intuitive explicit solution in the spirit of dependencies (18). At each time  $t$ , the solution is an eigenvector associated to a parameter  $W(t)$  with an instantaneous fitness  $\Lambda(W(t), R(t))$  associated to the same parameter  $W(t)$ . The function  $t \mapsto W(t)$  thus defined follows  $V(t)$  with a delay explicitly given by ODE (19).

A very useful consequence for the different applications is the existence of an invariant manifold for the growth-fragmentation equation with time-dependent parameters of the form (4). Let define the *eigenmanifold*

$$\mathcal{E} := \{Q \mathcal{U}(W; \cdot), (W, Q) \in (\mathbb{R}_+^*)^2\} \quad (21)$$

which is the union of all the positive eigenlines associated to a transport parameter  $W$ . Then Corollary 2.3 ensures that, under the assumptions of Theorem 2.1, any solution to Equation (1) with an initial distribution  $u_0 \in \mathcal{E}$  remains in  $\mathcal{E}$  for all time. Moreover the dynamics of such a solution reduces to the ODE (19) and this is the key point we use to tackle nonlinear problems.

For  $\nu \neq 1$ , the technique fails and we cannot obtain explicit solution with the method of Theorem 2.1. Nevertheless, we can still define  $W$  as the solution to ODE (19) and give properties of the functions defined as dilations of the eigenvector by (20). We obtain that the moments of these functions

satisfy equations that are similar to the ones verified by the moments of the solution to the growth-fragmentation equation. In the special case  $\nu = 0$ , and  $\gamma = 1$ , we even obtain the same equation. More precisely, if we denote, for  $\alpha \geq 0$ ,

$$\mathcal{M}_\alpha[W](t) := \int_0^\infty x^\alpha \mathcal{U}(W(t); x) e^{\int_0^t \Lambda(W(s), R(s)) ds} dx, \quad (22)$$

and

$$M_\alpha[u](t) := \int_0^\infty x^\alpha u(t, x) dx, \quad (23)$$

then we have the following result.

**Lemma 2.4.** *On the one hand, if  $W$  is a solution to Equation (19), then the moments  $\mathcal{M}_\alpha$  satisfy*

$$\dot{\mathcal{M}}_\alpha = \alpha \Lambda a_\alpha V \mathcal{M}_{\alpha+\nu-1} + (1-\alpha) \Lambda b_\alpha \mathcal{M}_{\alpha+\gamma} - \mu R \mathcal{M}_\alpha \quad (24)$$

with

$$a_\alpha := \frac{M_\alpha[\mathcal{U}]}{M_{\alpha+\nu-1}[\mathcal{U}]} \quad \text{and} \quad b_\alpha := \frac{M_\alpha[\mathcal{U}]}{M_{\alpha+\gamma}[\mathcal{U}]}.$$

On the other hand, if  $u$  is a solution to the fragmentation-drift equation, then the moments  $M_\alpha$  satisfy

$$\dot{M}_\alpha = \alpha \tau V M_{\alpha+\nu-1} + (c_\alpha - 1) \beta M_{\alpha+\gamma} - \mu R M_\alpha \quad (25)$$

with

$$c_\alpha := \int_0^1 z^\alpha \kappa(z) dz.$$

*Proof.* Thanks to a change of variable we can compute for all  $\alpha \geq 0$

$$\mathcal{M}_\alpha[W] = M_\alpha[\mathcal{U}] W^{k\alpha} e^{\int_0^t \Lambda(W(s), R(s)) ds}$$

so we have

$$\begin{aligned} \dot{\mathcal{M}}_\alpha &= k\alpha \frac{\dot{W}}{W} \mathcal{M}_\alpha + \Lambda(W, R) \mathcal{M}_\alpha \\ &= \alpha \Lambda W^{k\gamma-1} (V - W) \mathcal{M}_\alpha + \Lambda W^{k\gamma} \mathcal{M}_\alpha - \mu R \mathcal{M}_\alpha \\ &= \alpha \Lambda V W^{k(\nu-1)} \mathcal{M}_\alpha + (1-\alpha) \Lambda W^{k\gamma} \mathcal{M}_\alpha - \mu R \mathcal{M}_\alpha \\ &= \alpha \Lambda V a_\alpha \mathcal{M}_{\alpha+\nu-1} + (1-\alpha) \Lambda b_\alpha \mathcal{M}_{\alpha+\gamma} - \mu R \mathcal{M}_\alpha. \end{aligned}$$

Integrating Equation (1) against  $x^\alpha dx$  we obtain by integration by part and thanks to the Fubini theorem

$$\begin{aligned} \frac{d}{dt} \int x^\alpha u(t, x) dx &= \tau V \int x^\alpha \partial_x (x^\nu u(t, x)) dx - \mu R \int x^\alpha u(t, x) dx \\ &\quad - \beta \int x^{\alpha+\gamma} u(t, x) dx + \beta \int_0^\infty x^\alpha \int_x^\infty y^{\gamma-1} \kappa\left(\frac{x}{y}\right) dy dx \\ &= \alpha \tau V \int x^{\alpha+\nu-1} u(t, x) dx - \mu R \int x^\alpha u(t, x) dx \\ &\quad - \beta \int x^{\alpha+\gamma} u(t, x) dx + \beta \int_0^\infty y^{\alpha+\gamma} \int_0^y \frac{x^\alpha}{y^\alpha} \kappa\left(\frac{x}{y}\right) \frac{dx}{y} dy \\ &= \alpha \tau V M_{\alpha+\nu-1} + (c_\alpha - 1) \beta M_{\alpha+\gamma} - \mu R M_\alpha. \end{aligned}$$

□

In the particular case  $\nu = 0$ ,  $\gamma = 1$  and  $\kappa$  symmetric, we can compute the constants  $a_\alpha$ ,  $b_\alpha$  and  $c_\alpha$  for  $\alpha = 1$  or  $\alpha = 2$ . A consequence is the useful result given in the following corollary.

**Corollary 2.5.** *In the case when  $\nu = 0$ ,  $\gamma = 1$  and  $\kappa$  symmetric, the zero and first moments  $(\mathcal{M}_0, \mathcal{M}_1)$  and  $(M_0, M_1)$  are both solution to*

$$\begin{pmatrix} \dot{U} \\ \dot{P} \end{pmatrix} = \begin{pmatrix} -\mu R & \beta \\ \tau V & -\mu R \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix}. \quad (26)$$

This Corollary allows in Section 6 to compare the Perron and Floquet eigenvalues not only for  $\nu = 1$  but also for  $\nu = 0$ ,  $\gamma = 1$ . We do not have in this case a particular solution to the growth-fragmentation equation as in Corollary 2.3, but a particular solution of the reduced ODE system (26).

*Proof.* For  $\kappa$  symmetric, we have already seen that  $c_0 = n_0 = 2$ . Together with Assumption (3) which gives  $c_1 = 1$ , we conclude that  $(M_0, M_1)$  is solution to Equation (26).

Integrating Equation (16) against  $dx$  and  $x dx$  we obtain

$$\Lambda = \beta \int x \mathcal{U}(x) dx \quad \text{and} \quad \Lambda \int x \mathcal{U}(x) dx = \tau.$$

It allows to compute

$$\int x \mathcal{U}(x) dx = \sqrt{\frac{\tau}{\beta}} \quad \text{and} \quad \Lambda = \sqrt{\tau\beta}.$$

Thus  $a_1 = \sqrt{\frac{\tau}{\beta}}$  and  $b_0 = \sqrt{\frac{\beta}{\tau}}$ , and  $(\mathcal{M}_0, \mathcal{M}_1)$  satisfies Equation (26) thanks to Lemma 2.4.  $\square$

### 2.3 General Relative Entropy: Attractivity of the Invariant Manifold

The existence of the invariant manifold  $\mathcal{E}$  is useful to obtain particular solutions to nonlinear growth-fragmentation equations. But what happens when the initial distribution  $u_0$  does not belong to this manifold ? The GRE method ensures that  $\mathcal{E}$  is attractive in a sense to be defined.

The GRE method requires to consider the adjoint growth-fragmentation equation

$$-\frac{\partial}{\partial t} \psi(t, x) = \tau(t, x) \frac{\partial}{\partial x} \psi(t, x) - \mu(t, x) \psi(t, x) + (\mathcal{F}^* \psi)(t, x) \quad (27)$$

where  $\mathcal{F}^*$  is the adjoint fragmentation operator

$$(\mathcal{F}^* \psi)(t, x) := \int_0^x b(t, x, y) \psi(t, y) dy - \beta(t, x) \psi(t, x).$$

If  $u$  and  $v$  are two solutions to Equation (1) and  $\psi$  is a solution to Equation (27), then we have for any function  $H : \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \psi(t, x) v(t, x) H\left(\frac{u(t, x)}{v(t, x)}\right) dx &= - \int_0^\infty \int_y^\infty b(t, y, x) \psi(t, x) v(t, y) \\ &\times \left[ H\left(\frac{u(t, x)}{v(t, x)}\right) - H\left(\frac{u(t, y)}{v(t, y)}\right) + H'\left(\frac{u(t, x)}{v(t, x)}\right) \left(\frac{u(t, y)}{v(t, y)} - \frac{u(t, x)}{v(t, x)}\right) \right] dx dy. \end{aligned}$$



When  $H$  is convex, the right hand side is nonpositive and we obtain a nonincreasing quantity called GRE.

In the case of time-independent coefficients, we can chose for  $v$  a solution of the form  $\mathcal{U}(x)e^{\Lambda t}$ . Then, to apply the GRE method, we need a solution to the adjoint equation and such solutions are given by solving the adjoint Perron eigenvalue problem

$$\begin{cases} \Lambda\phi(x) = \tau(x)\frac{\partial}{\partial x}(\phi(x)) - \mu(x)\phi(x) + (\mathcal{F}^*\phi)(x), & x \geq 0, \\ \phi(x) \geq 0, & \int_0^\infty \phi(x)\mathcal{U}(x)dx = 1. \end{cases} \quad (28)$$

Such a problem is usually solved together with the direct problem (16) and the first eigenvalue  $\Lambda$  is the same for the two ones (see [23, 46, 51]). Then the GRE ensures that any solution  $u$  to the growth-fragmentation equation behaves asymptotically like  $\mathcal{U}(x)e^{\Lambda t}$ . More precisely it is proved in [49, 51] under general assumptions that

$$\lim_{t \rightarrow \infty} \|\varrho_0^{-1}u(t, \cdot)e^{-\Lambda t} - \mathcal{U}\|_{L^p(\mathcal{U}^{1-p}\phi dx)} = 0, \quad \forall p \geq 1. \quad (29)$$

where  $\varrho_0 = \int u_0(y)\phi(y) dy$  with  $u_0(x) = u(t=0, x)$ .

Under the assumptions of Theorem 2.1 and for  $p = 1$ , this convergence result can be interpreted as the attractivity of the invariant manifold  $\mathcal{E}$  in  $L^1(\phi dx)$  with the distance

$$d(u, \mathcal{E}) := \inf_{\mathcal{U} \in \mathcal{E}} \|\varrho^{-1}u - \mathcal{U}\|_{L^1(\phi(x) dx)}$$

where  $\varrho := \int u(y)\phi(y) dy$ . Consider, for  $V(t) \geq 0$ , a solution  $W$  to

$$\dot{W} = \frac{W}{k}(\tau - W)$$

with  $W(0) = 1$ . First we have  $\dot{W} \geq -\frac{1}{k}W^2$ , so  $W \geq \frac{1}{1+\frac{t}{k}}$  and  $h \geq k \ln(1 + \frac{t}{k})$ . Thus  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is one to one and we can define from a solution  $u(t, x)$  the function

$$v(h(t), x) := W^k(t) u(t, W^k(t)x) e^{-\int_0^t (W(s) - \mu R(s)) ds} \quad (30)$$

which satisfies  $v(0, x) = u(0, x) = u_0(x)$ . Then, using Theorem 2.1,  $v(t, x)$  is a solution to

$$\frac{\partial}{\partial t}v(t, x) = -\frac{\partial}{\partial x}(xv(t, x)) - v(t, x) + (\mathcal{F}_\gamma v)(t, x)$$

so we have

$$\int_0^\infty v(t, x)\phi(x) dx = \int_0^\infty v(t=0, x)\phi(x) dx = \varrho_0$$

and

$$\lim_{t \rightarrow \infty} \|\varrho_0^{-1}v(t, \cdot) - \mathcal{U}\|_{L^1(\phi(x) dx)} = 0.$$

For  $\nu = 1$ ,  $\phi$  is linear (see examples in [23]), so we can compute from (30)

$$\varrho(t) = \int u(t, y)\phi(y) dy = \varrho_0 W^k(t) e^{\int_0^t (W(s) - \mu R(s)) ds}$$

and

$$d(u(t, \cdot), \mathcal{E}) \leq \|\varrho^{-1}(t)u(t, \cdot) - \mathcal{U}\| = \|\varrho_0^{-1}v(t, \cdot) - \mathcal{U}\| \rightarrow 0. \quad (31)$$

The exponential decay in (29) is proved in [52, 39] for  $p = 1$  and for a constant fragmentation rate  $\beta(x) = \beta$ . It is also proved in [8] for powerlaw parameters in the norm corresponding to  $p = 2$  and this is the case we are interested in. Assume that the coefficients satisfy (5), (6) and (7) and assume also that the fragmentation kernel is upper and lower bounded

$$\exists \underline{\kappa}, \bar{\kappa} > 0, \quad \forall z \in [0, 1], \quad \underline{\kappa} \leq \kappa(z) \leq \bar{\kappa}. \quad (32)$$

Then a spectral gap result is proved in [8] for  $\nu = 1$  in  $L^2(\mathcal{U}^{-1}\phi dx)$  and then the result is extended to bigger spaces thanks to a general method for spectral gap.

**Theorem [8].** *Under Assumption (5) with  $\nu = 1$ , Assumptions (6) and (32), and Assumption (7) with  $\gamma \in (0, 2]$ , there exists  $\bar{r} \geq 3$  and for any  $a \in (0, \gamma)$  and any  $r \geq \bar{r}$  there exists  $C_{a,r}$  such that for any  $u_0 \in \mathcal{H} := L^2(\theta)$ ,  $\theta(x) = x + x^r$ , there holds*

$$\forall t > 0, \quad \|\varrho_0^{-1}u(t, \cdot)e^{-\Lambda t} - \mathcal{U}\|_{\mathcal{H}} \leq C_{a,r} \|\varrho_0^{-1}u_0 - \mathcal{U}\|_{\mathcal{H}} e^{-at}. \quad (33)$$

This result is very useful for Applications 1 and 2 because  $L^2(\theta) \subset L^1(x^p)$  for  $r \geq 2p + 1$ . Moreover the exponential decay allows to prove exponential stability results for Equation (8) when  $\kappa$  is constant (see Section 3).

### 3 Nonlinear Drift Term: Convergence and Stability

Consider the nonlinear growth-fragmentation equation (8) where the transport term depends on the  $p$ -moment of the solution itself. This dependency may represent the influence of the total population of individuals on the growth process of each individual. We study the long-time asymptotic behavior of the solutions in the positive cone  $\mathcal{H}^+$  with the weight  $\theta(x) = x + x^r$  for

$$r \geq \max(\bar{r}, 2p + 1). \quad (34)$$

We prove that there is always convergence to a steady state, provided that the function  $f$  is less than  $\mu$  at the infinity. This result is precised in Theorem 3.1 which immediately leads to Theorem 1.1 and the stability of the steady states is given in Theorem 1.2. We use the notation  $M_p$  for  $M_p[\mathcal{U}] = \int x^p \mathcal{U}(x) dx$ .

**Theorem 3.1.** *Assume that  $f$  is a continuous function on  $[0, +\infty)$  which satisfies Assumption (9), that  $\gamma \in (0, 2]$  and that the fragmentation kernel  $\kappa$  satisfies Assumption (32). Then the nontrivial steady states of Equation (8) write*

$$\frac{I_\infty}{M_p} \mu^{-kp} \mathcal{U}(\mu; \cdot) \quad \text{with} \quad I_\infty \in \mathcal{N},$$

and for any solution  $u$ , there exists  $I_\infty \in \mathcal{N} \cup \{0\}$  such that

$$\lim_{t \rightarrow \infty} \left\| u(t, \cdot) - \frac{I_\infty}{M_p} \mu^{-kp} \mathcal{U}(\mu; \cdot) \right\|_{\mathcal{H}} = 0. \quad (35)$$

*Proof of Theorem 3.1. First step:*  $u_0 \in \mathcal{E}$ .

Consider an initial distribution  $u_0 \in \mathcal{E}$  defined in (21), then there exist  $W_0 > 0$  and  $Q_0 \geq 0$  such that

$$u_0(x) = Q_0 \mathcal{U}(W_0 \mu; x).$$

Let  $u(t, x)$  be the solution to Equation (8) and define  $W$  as the solution to

$$\begin{cases} \dot{W} &= \frac{W}{k} \left( f \left( \int_0^\infty x^p u(t, x) dx \right) - \mu W \right), \\ \dot{W}(0) &= W_0. \end{cases} \quad (36)$$

Then Corollary (2.3) ensures that

$$\forall t, x \geq 0, \quad u(t, x) = Q_0 \mathcal{U}(W(t) \mu; x) e^{\mu \int_0^t (W(s) - 1) ds}$$

and so we have

$$\int_0^\infty x^p u(t, x) dx = Q_0 W^{kp} \mu^{kp} \left( \int_0^\infty x^p \mathcal{U}(x) dx \right) e^{\mu \int_0^t (W(s) - 1) ds}.$$

Now defining

$$Q(t) := Q_0 e^{\mu \int_0^t (W(s) - 1) ds},$$

we obtain a system of ODEs equivalent to Equation (8) in  $\mathcal{E}$

$$\begin{cases} \dot{W} &= \frac{W}{k} \left( f_p(W^{kp} Q) - \mu W \right), \\ \dot{Q} &= \mu Q(W - 1), \end{cases} \quad (37)$$

with the notation

$$f_p(I) := f \left( I \mu^{kp} \int_0^\infty x^p \mathcal{U}(x) dx \right),$$

and proving convergence of  $u$  is equivalent to prove convergence of  $(W, Q)$ . To study System (37), we change the unknown to  $Z := W^{kp} Q$ . Then  $(W, Z)$  is solution to

$$\begin{cases} \dot{W} &= \frac{W}{k} (f_p(Z) - \mu W), \\ \dot{Z} &= pZ (f_p(Z) - \mu W) + \mu Z(W - 1), \end{cases} \quad (38)$$

and the positive steady states satisfy  $W_\infty = \mu^{-1} f_p(Z_\infty) = 1$ . If  $p = 1$ ,  $Z$  is solution to  $\dot{Z} = Z(f_1(Z) - \mu)$  so for  $Z_0 \geq 0$ ,  $Z$  converges to  $Z_\infty = \frac{I_\infty}{\mu^{kp} M_p}$  with  $I_\infty \in \mathcal{N} \cup \{0\}$ . When  $Z_\infty > 0$ , then  $W \rightarrow 1$  because  $f_1(Z) \rightarrow \mu$ .

In the case when  $p \neq 1$ , we write System (38) as

$$\begin{cases} \dot{W} &= -\frac{1}{k} W(\mu - f_p(Z)) - \frac{\mu}{k} W(W - 1) \\ \dot{Z} &= -pZ(\mu - f_p(Z)) - (p - 1)\mu Z(W - 1), \end{cases} \quad (39)$$

and we exhibit a Lyapunov functional for this equation. We define

$$G(W) := W - 1 - \ln(W)$$

and

$$F(Z) := \int_1^Z (\mu - f_p(z)) \frac{dz}{z},$$

then we look for a parameter  $\alpha$  such that  $k\mu G(W) + \alpha^2 F(Z)$  is a Lyapunov functional for Equation (39). We have

$$\frac{d}{dt} (k\mu G(W) + \alpha^2 F(Z)) = -\mu^2(W-1)^2 - \alpha^2 p(\mu - f_p(Z))^2 - \mu(1 + \alpha^2(p-1))(W-1)(\mu - f_p(Z))$$

so, if we find  $\alpha$  such that  $1 + \alpha^2(p-1) = 2\alpha\sqrt{p}$ , we will have

$$\frac{d}{dt} (k\mu G(W) + \alpha^2 F(Z)) = -(\mu(W-1) + \alpha\sqrt{p}(\mu - f_p(Z)))^2 \leq 0.$$

For  $p \neq 1$ , there are two roots to the binomial

$$(p-1)\alpha^2 - 2\sqrt{p}\alpha + 1 = 0$$

which are

$$\alpha = \frac{\sqrt{p} \pm 1}{p-1} = \frac{1}{\sqrt{p} \pm 1}.$$

For these two values, the dissipation is nonpositive but we want to have negativity outside of the steady states in order to conclude to convergence. We use a combination of this two values and the following lemma.

**Lemma 3.2.** *For any  $\alpha \neq 1$ , there exists  $\omega > 0$  such that*

$$\forall a, b, \quad (a+b)^2 + (a+\alpha b)^2 \geq \omega(a^2 + b^2). \quad (40)$$

*Proof.* We prove the inequality

$$(a+b)^2 + (a+\alpha b)^2 \geq f(\alpha-1)(a^2 + b^2) \quad (41)$$

where

$$f(x) = \frac{4 - 2\sqrt{4+x^2} + x^2}{2 + \frac{x}{2}\sqrt{4+x^2} + \frac{x^2}{2}}.$$

This function is positive except at  $x = 0$  where  $f(0) = 0$ . Its maximum is  $f(-2) = 2$ .

To prove (41), we define  $\beta := \frac{\alpha}{b}$  and we prove that

$$f(1-\alpha) = \min_{\beta} \left( \frac{(\beta+1)^2 + (\beta+\alpha)^2}{\beta^2 + 1} \right).$$

Define

$$g_{\alpha}(\beta) := \frac{(\beta+1)^2 + (\beta+\alpha)^2}{\beta^2 + 1},$$

compute

$$g'_\alpha(\beta) = \frac{(1 + \alpha) - (\alpha^2 - 1)\beta - (1 + \alpha)\beta^2}{(1 + \beta^2)^2},$$

and we find that the minimum of  $g$  is reached for

$$\beta_\alpha = -\frac{1}{2} \left( \alpha - 1 + \sqrt{4 + (\alpha - 1)^2} \right).$$

As a consequence, for all  $a \in \mathbb{R}$  and  $b \neq 0$ , we have

$$(a + b)^2 + (a + \alpha b)^2 \geq g_\alpha(\beta_\alpha)(a^2 + b^2)$$

and  $g_\alpha(\beta_\alpha) = f(1 - \alpha)$ . This is still true for  $b = 0$  by passing to the limit.  $\square$

Denoting  $\alpha_+ = \frac{1}{\sqrt{p+1}}$  and  $\alpha_- = \frac{1}{\sqrt{p-1}}$ , we define  $L(W, Z) := 2k\mu G(W) + (\alpha_+^2 + \alpha_-^2)F(Z)$ . Then Lemma 3.2 ensures the existence of  $\omega > 0$  such that

$$\begin{aligned} \frac{d}{dt}L(W(t), Z(t)) &= -(\mu(W - 1) + \alpha_+\sqrt{p}(\mu - f_p(Z)))^2 - (\mu(W - 1) + \alpha_-\sqrt{p}(\mu - f_p(Z)))^2 \\ &\leq -\omega(\mu^2(W - 1)^2 + \alpha_+^2 p(\mu - f_p(Z))^2) := -D(W, Z), \end{aligned} \quad (42)$$

and  $D(W, Z)$  is positive outside of the steady states. Moreover Assumption (9) ensures that  $L$  and  $D$  are coercive in the sense that  $L(W, Z) \rightarrow +\infty$  and  $D(W, Z) \rightarrow +\infty$  when  $\|(W, Z)\| \rightarrow +\infty$ . So  $L$  is a Lyapunov functional for System (39) and we can conclude to the convergence of the solution to a steady state. If  $f(0) > \mu$ ,  $L(W, Z) \rightarrow +\infty$  if  $W$  or  $Z$  tends to 0, so for any  $(W_0 > 0, Z_0 > 0)$  the solution  $(W, Z)$  converges to a critical point of  $L$ , namely  $(1, \frac{I_\infty}{\mu^{kp}M_p})$  with  $I_\infty \in \mathcal{N}$ . If  $f(0) < \mu$ , then for any  $\bar{W} > 0$  we have that  $L(W, Z) \xrightarrow{(W,Z) \rightarrow (\bar{W}, 0)} -\infty$ . So either  $(W, Z)$  converges to  $(1, \frac{I_\infty}{\mu^{kp}M_p})$  with  $I_\infty \in \mathcal{N}$ , or  $Z \rightarrow 0$ . To conclude to the convergence in  $\mathcal{H}$ , we write

$$\|u(t, \cdot) - Z_\infty \mathcal{U}(\mu; \cdot)\| = \int_0^\infty (Q(t)\mathcal{U}(W(t)\mu; x) - Z_\infty \mathcal{U}(\mu; x))^2 (x + x^r) dx$$

and we use dominated convergence. We know by Theorem 1 in [23] that under Assumption (32) and for  $\nu = 1$ ,  $x^\alpha \mathcal{U}(x)$  is bounded in  $\mathbb{R}^+$  for all  $\alpha \geq 0$ , so it ensures that the integrand is dominated by an integrable function. Then the ponctual convergence is given by the convergence of  $(W, Z)$ , so convergence (35) occurs.

**Second step: general initial distribution  $u_0$ .**

Departing from  $u$  a solution to this Equation (8) not necessarily in  $\mathcal{E}$ , we define as in Section 2.3

$$v(h(t), x) := W(t)^k u(t, W(t)^k x) e^{\mu(t-h(t))}$$

with

$$\dot{W} = \frac{W}{k} \left( f \left( \int x^p u \right) - \mu W \right)$$

and

$$\dot{h} = W.$$

We recall that in this case  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is one to one. We choose the initial values  $W(0) = 1$  and  $h(0) = 0$  to have  $v(0, x) = u(0, x) = u_0(x)$ . Then  $v(t, x)$  is a solution to

$$\frac{\partial}{\partial t} v(t, x) = -\mu \frac{\partial}{\partial x} (xv(t, x)) - \mu v(t, x) + \mathcal{F}_\gamma v(t, x) \quad (43)$$

and, thanks to the GRE, we conclude that

$$v(t, x) \xrightarrow[t \rightarrow \infty]{} \varrho_0 \mathcal{U}(\mu; x)$$

where

$$\varrho_0 = \int_0^\infty \phi(\mu; x) v_0(x) dx = \int_0^\infty \phi(\mu; x) u_0(x) dx$$

and so

$$\int x^p u(t, x) dx \underset{t \rightarrow \infty}{\sim} \varrho_0 \mu^{kp} \left( \int x^p \mathcal{U}(x) dx \right) W(t)^{kp} e^{\mu(h(t)-t)} dx.$$

As a consequence it holds

$$\dot{W}(t) \underset{t \rightarrow \infty}{\sim} \frac{W}{k} \left( f_p \left( W^{kp} Q \right) - \mu W \right)$$

with  $Q(t) = \varrho_0 e^{\mu(h(t)-t)}$  which satisfies the equation

$$\dot{Q} = \mu Q(W - 1).$$

The interpretation of this is that System (36) represents asymptotically the dynamics of the solutions to Equation (8). More rigorously, define

$$\varepsilon(t) := \frac{\int x^p u(t, x) dx}{\mu^{kp} M_p Q(t) W^{kp}(t)} - 1. \quad (44)$$

Then we have

$$\dot{W}(t) = \frac{W}{k} \left( f_p \left( W^{kp} Q(1 + \varepsilon(t)) \right) - \mu W \right)$$

and, using the Cauchy-Schwarz inequality and the exponential decay theorem of [8],

$$\begin{aligned} |\varepsilon(t)| &= \frac{W^{-kp} e^{\mu(t-h(t))}}{\mu^{kp} M_p} \left| \int (\varrho_0^{-1} u(t, x) - \mathcal{U}(\mu; x) W^{kp} e^{\mu(h(t)-t)}) x^p dx \right| \\ &= \frac{1}{\mu^{kp} M_p} \left| \int (\varrho_0^{-1} v(h(t), x) - \mathcal{U}(\mu; x)) x^p dx \right| \\ &\leq \frac{1}{\mu^{kp} M_p} \left( \int |\varrho_0^{-1} v(h(t), x) - \mathcal{U}(\mu; x)|^2 (\phi(x) + x^r) dx \right)^{\frac{1}{2}} \left( \int \frac{x^{2p}}{\phi(x) + x^r} dx \right)^{\frac{1}{2}} \\ &\leq \frac{C_p}{\mu^{kp} M_p} \|\varrho_0^{-1} u_0 - \mathcal{U}(\mu; \cdot)\| e^{-ah(t)}. \end{aligned} \quad (45)$$

The function  $\frac{x^{2p}}{x+x^r}$  is integrable under Assumption (34). Finally the dynamics of the solution  $u$  to Equation (8) is given by

$$\begin{cases} \dot{W} &= -\frac{1}{k} W (\mu - f_p((1 + \varepsilon)Z)) - \frac{\mu}{k} W(W - 1) \\ \dot{Z} &= -pZ (\mu - f_p((1 + \varepsilon)Z)) - (p - 1)\mu Z(W - 1), \end{cases} \quad (46)$$

where  $Z = W^{kp}Q$ , and  $\varepsilon(t) \xrightarrow[t \rightarrow \infty]{} 0$ . Now we look what becomes the Lyapunov functional of the first step for this system and we obtain

$$\frac{d}{dt}L(W, Z) \leq -D(W, Z) + \underbrace{(\mu(W - 1) + (\alpha_+^2 + \alpha_-^2)p(\mu - f_p(Z)))(f_p((1 + \varepsilon)Z) - f_p(Z))}_{:=E(W, Z, \varepsilon)}. \quad (47)$$

Thanks to Assumption (9), we know that  $f_p$  is bounded, so  $W$  which is solution to

$$\dot{W} = \frac{W}{k}(f_p((1 + \varepsilon)Z) - \mu W)$$

is also bounded. Thus  $E(W, Z, \varepsilon)$  is bounded and, because  $L$  and  $D$  are coercive, the trajectory  $(W, Z)$  is bounded. Moreover  $E(W(t), Z(t), \varepsilon(t)) \xrightarrow[t \rightarrow \infty]{} 0$  because  $\varepsilon(t) \rightarrow 0$ , so  $(W, Z)$  converges to a steady state  $(1, Z_\infty)$ . Finally we write

$$\|u(t, \cdot) - Z_\infty \mathcal{U}(\mu; \cdot)\| \leq \|u(t, \cdot) - Q(t)\mathcal{U}(W(t)\mu; \cdot)\| + \|Q(t)\mathcal{U}(W(t)\mu; \cdot) - Z_\infty \mathcal{U}(\mu; \cdot)\|$$

and we know from the first step that  $\|Q(t)\mathcal{U}(W(t)\mu; \cdot) - Z_\infty \mathcal{U}(\mu; \cdot)\| \rightarrow 0$ . We treat the other term thanks to the spectral gap theorem of [8]

$$\begin{aligned} \|u(t, \cdot) - Q(t)\mathcal{U}(W(t)\mu; \cdot)\|_{\mathcal{H}} &= Q \left( \int |\varrho_0^{-1}v(h(t), x) - \mathcal{U}(\mu; x)|^2 (x + W^{(r-1)k}x^r) dx \right)^{\frac{1}{2}} \\ &\leq C \|\varrho_0^{-1}v(h(t), \cdot) - \mathcal{U}(\mu; \cdot)\| \\ &\leq C \|\varrho_0^{-1}u_0 - \mathcal{U}(\mu; \cdot)\| e^{-ah(t)}. \end{aligned} \quad (48)$$

So  $\|u(t, \cdot) - Q(t)\mathcal{U}(W(t)\mu; \cdot)\| \rightarrow 0$  because  $h(t) \rightarrow +\infty$ , and the proof is complete.  $\square$

**Proof of Theorem 1.2. Stability of the trivial steady state.**

We start with the stability of the zero steady state when  $f(0) < \mu$  and  $p \geq 1$ . For this we integrate Equation (8) against  $x^p$  and find

$$\frac{d}{dt} \left( \int x^p u(t, x) dx \right) \leq \left( f \left( \int x^p u(t, x) dx \right) - \mu \right) \left( \int x^p u(t, x) dx \right)$$

thanks to the mass conservation assumption (3). Thus  $\int x^p u(t, x) dx$  is a decreasing function if  $f(M_p[u_0]) < \mu$ . Then we integrate against  $x + x^r$  for any  $r \geq 1$  and we obtain

$$\frac{d}{dt} \left( \int u(t, x) (x + x^r) dx \right) \leq \left( f \left( \int x^p u(t = 0, x) dx \right) - \mu \right) \left( \int u(t, x) (x + x^r) dx \right)$$

which ensures the exponential convergence

$$\|u(t, \cdot)\|_{\mathcal{H}} \leq \|u_0\|_{\mathcal{H}} e^{(f(M_p[u_0]) - \mu)t}.$$

For  $r \geq p$ , we have  $M_p[u_0] \leq C\|u_0\|_{\mathcal{H}}$ , so for  $\|u_0\|$  small enough, we have  $f(M_p[u_0]) < \mu$  and the exponential convergence occurs.

When  $f(0) > \mu$ , we have seen in the proof of Theorem 3.1 that  $L(W, Z) \rightarrow +\infty$  when  $W$  or  $Z$  tends to zero. So the trivial steady state is unstable.

### Stability of nontrivial steady states.

Let  $(W_\infty, Z_\infty)$  be a positive steady state to System (39). We want to prove that  $Z_\infty \mathcal{U}(\mu; \cdot)$  is locally asymptotically stable. Since Theorem 3.1 ensures the convergence of any solution to Equation (8) toward a steady state, it only remains to prove the local stability of  $Z_\infty \mathcal{U}(\mu; \cdot)$ , namely

$$\forall \rho_1 > 0, \exists \rho_2 > 0, \quad \|u_0 - Z_\infty \mathcal{U}(\mu; \cdot)\| < \rho_2 \Rightarrow \forall t > 0, \|u(t, \cdot) - Z_\infty \mathcal{U}(\mu; \cdot)\| < \rho_1.$$

We have already seen that

$$\|u(t, \cdot) - Z_\infty \mathcal{U}(\mu; \cdot)\| \leq C \|\varrho_0^{-1} u_0 - \mathcal{U}(\mu; \cdot)\| + \|Q(t) \mathcal{U}(W(t); \cdot) - Z_\infty \mathcal{U}(\mu; \cdot)\|$$

with  $\|Q \mathcal{U}(W; \cdot) - Z_\infty \mathcal{U}(\mu; \cdot)\| \rightarrow 0$  when  $(W, Z) \rightarrow (W_\infty, Z_\infty)$ .

Let first treat the term  $\|\varrho_0^{-1} u_0 - \mathcal{U}(\mu; \cdot)\|$ . We have

$$\begin{aligned} |\varrho_0 - Z_\infty| &= \left| \int (u_0 - Z_\infty \mathcal{U}(\mu; x)) \phi(\mu; x) dx \right| \\ &\leq \mu^{-k} \left( \int (u_0 - Z_\infty \mathcal{U}(\mu; x))^2 (x + x^r) dx \right)^{\frac{1}{2}} \left( \int \frac{x^2}{x + x^r} dx \right)^{\frac{1}{2}} \\ &= C \|u_0 - Z_\infty \mathcal{U}(\mu; \cdot)\| \end{aligned} \tag{49}$$

and then

$$\begin{aligned} \|\varrho_0^{-1} u_0 - \mathcal{U}(\mu; \cdot)\| &\leq \varrho_0^{-1} \|u_0 - \frac{I_\infty}{\mu^{kp} M_p} \mathcal{U}(\mu; \cdot)\| + \varrho_0^{-1} |\varrho_0 - Z_\infty| \|\mathcal{U}(\mu; \cdot)\| \\ &\leq \frac{C}{\varrho_0} \|u_0 - Z_\infty \mathcal{U}(\mu; \cdot)\| \\ &\leq \frac{C}{Z_\infty - |\varrho_0 - Z_\infty|} \|u_0 - Z_\infty \mathcal{U}(\mu; \cdot)\| \\ &\leq \frac{C \|u_0 - Z_\infty \mathcal{U}(\mu; \cdot)\|}{Z_\infty - C \|u_0 - Z_\infty \mathcal{U}(\mu; \cdot)\|} \end{aligned} \tag{50}$$

so  $\|\varrho_0^{-1} u_0 - \mathcal{U}(\mu; \cdot)\|$  is small for  $\|u_0 - Z_\infty \mathcal{U}(\mu; \cdot)\|$  small enough.

Now let turn to the term  $\|Q \mathcal{U}(W; \cdot) - Z_\infty \mathcal{U}(\mu; \cdot)\|$ . Since it tends to zero when  $(W, Z)$  tends to  $(W_\infty, Z_\infty)$ , it is sufficient to prove that  $(W_\infty, Z_\infty)$  is stable for System (46). For  $\eta > 0$ , denote by  $\mathcal{V}_\eta(W_\infty, Z_\infty)$  the connex component of  $\{(W, Z), L(W, Z) < L(W_\infty, Z_\infty) + \eta\}$  which contains  $(W_\infty, Z_\infty)$ . Since  $f'(I_\infty) < 0$ ,  $(W_\infty, Z_\infty)$  is a strict local minimum of  $L$  so we have, denoting  $B(X, \rho) = \{Y \in \mathbb{R}^2, \|X - Y\| < \rho\}$ ,

$$\forall \rho > 0, \exists \eta > 0, \quad \mathcal{V}_\eta(W_\infty, Z_\infty) \subset B((W_\infty, Z_\infty), \rho)$$

and reciprocally

$$\forall \eta > 0, \exists \rho > 0, \quad B((W_\infty, Z_\infty), \rho) \subset \mathcal{V}_\eta(W_\infty, Z_\infty).$$

So it is sufficient to have the local stability of  $(W_\infty, Z_\infty)$  to prove that  $\mathcal{V}_\eta(W_\infty, Z_\infty)$  is stable. This is true for System (39) since in this case  $L$  is a Lyapunov functional. Then, by continuity of  $f$ , there



exists  $\varepsilon_\eta$  such that  $\mathcal{V}_\eta(W_\infty, Z_\infty)$  remains stable for System (46) if  $|\varepsilon(t)| < \varepsilon_\eta$  for all  $t > 0$ . But we know from (45) that  $|\varepsilon(t)| \leq C\|\varrho_0^{-1}u_0 - \mathcal{U}(\mu; \cdot)\|$  and from (50) that  $\|\varrho_0^{-1}u_0 - \mathcal{U}(\mu; \cdot)\|$  is small for  $\|u_0 - Z_\infty\mathcal{U}(\mu; \cdot)\|$  small enough. Finally  $\|Q\mathcal{U}(W\mu; \cdot) - Z_\infty\mathcal{U}(\mu; \cdot)\|$  is small for  $\|u_0 - Z_\infty\mathcal{U}(\mu; \cdot)\|$  small enough and the local asymptotical stability of the nontrivial steady states is proved.

Now assume that  $\kappa \equiv 2$  and prove the local exponential stability of  $Z_\infty\mathcal{U}(\mu; \cdot)$ . In this case we have (see examples in [23]) the explicit formula

$$\mathcal{U}(x) = C e^{-\frac{\beta}{\gamma}x^\gamma} \quad (51)$$

and thanks to this we can estimate the quantity  $\|Q\mathcal{U}(W\mu; \cdot) - Z_\infty\mathcal{U}(\mu; \cdot)\|$ . We have

$$\begin{aligned} \|Q\mathcal{U}(W\mu; \cdot) - Z_\infty\mathcal{U}(\mu; \cdot)\|^2 &= \int |Q\mathcal{U}(W^k\mu; x) - Z_\infty\mathcal{U}(\mu; x)|^2 (x + x^r) dx \\ &\leq |Z - Z_\infty|^2 W^{-2kp} \int \mathcal{U}(W^k\mu; x)^2 (x + x^r) dx \quad (i) \\ &\quad + |W^{-k(p+1)} - 1|^2 Z_\infty^2 \int \mathcal{U}(W^{-k}\mu^{-k}x)^2 (x + x^r) dx \quad (ii) \\ &\quad + Z_\infty^2 \int |\mathcal{U}(W^{-k}\mu^{-k}x) - \mathcal{U}(\mu^{-k}x)|^2 (x + x^r) dx \quad (iii) \end{aligned}$$

and we prove exponential decay of (i), (ii) and (iii) in a neighbourhood of  $(W_\infty, Z_\infty)$ . We have thanks to the L'Hôpital rule

$$F(Z) \underset{Z \rightarrow Z_\infty}{\sim} \frac{1}{-2Z_\infty f'(Z_\infty)} (\mu - f(Z))^2 \quad (52)$$

and

$$G(W) \underset{W \rightarrow W_\infty}{\sim} \frac{1}{2W_\infty} (W - 1)^2, \quad (53)$$

so the following local “entropy - entropy dissipation” inequality holds

$$\exists C > 0, \rho > 0, \quad \forall (W, Z) \in B((W_\infty, Z_\infty), \rho), \quad L(W, Z) \leq CD(W, Z). \quad (54)$$

Fix such a  $\rho$  and fix  $\eta > 0$  such that  $\mathcal{V}_\eta(W_\infty, Z_\infty) \subset B((W_\infty, Z_\infty), \rho)$ . Consider  $\|u_0 - Z_\infty\mathcal{U}(\mu; \cdot)\|$  small enough so that  $|\varepsilon(t)|$  remains smaller than  $\varepsilon_\eta$  for all time. Then  $\mathcal{V}_\eta(W_\infty, Z_\infty)$  is stable for the dynamics of System (46). Now look at the term  $E(W(t), Z(t), \varepsilon(t))$  for  $(W, Z)$  solution to System (46) in this stable neighbourhood. It satisfies

$$\begin{aligned} E &= \mu(W - 1)(f_p((1 + \varepsilon)Z) - f_p(Z)) + (\alpha_+^2 + \alpha_-^2)p(\mu - f_p(Z))(f_p((1 + \varepsilon)Z) - f_p(Z)) \\ &\leq \frac{\omega}{2}\mu^2(W - 1)^2 + \frac{\omega}{2}\alpha_+^2 p(\mu - f_p(Z))^2 + \frac{1}{2\omega} \left( 1 + \left( \frac{\alpha_+^2 + \alpha_-^2}{\alpha_+} \right)^2 p \right) (f_p((1 + \varepsilon)Z) - f_p(Z))^2 \\ &= \frac{1}{2}D(W, Z) + C(f_p((1 + \varepsilon)Z) - f_p(Z))^2 \\ &\leq \frac{1}{2}D(W, Z) + C \sup_J |f'| \varepsilon^2 \end{aligned}$$

where  $J = [Z_\infty - \rho - \varepsilon_\eta, Z_\infty + \rho + \varepsilon_\eta]$ . As a consequence we have

$$\begin{aligned} \frac{d}{dt}L(W, Z) &\leq -\frac{1}{2}D(W, Z) + C\varepsilon^2 \\ &\leq -CL(W, Z) + C\|\varrho_0^{-1}u_0 - \mathcal{U}(\mu; \cdot)\|^2 e^{-2ah(t)} \end{aligned}$$

and so

$$L(W, Z) \leq L(W_0, Z_0)e^{-Ct} + C\|\varrho_0^{-1}u_0 - \mathcal{U}(\mu; \cdot)\|^2 e^{-2ah(t)}.$$

Then, thanks to the equivalences (52) and (53) and because  $h(t) \sim t$  when  $t \rightarrow \infty$ , there are  $b > 0$  and  $C > 0$  such that

$$(W - 1)^2 + (\mu - f_p(Z))^2 \leq C(\mu - f_p(Z_0))^2 e^{-2bt} + C\|\varrho_0^{-1}u_0 - \mathcal{U}(\mu; \cdot)\|^2 e^{-2bt}$$

and, because  $f'(I_\infty) \neq 0$ , it holds

$$(W - 1)^2 + (Z - Z_\infty)^2 \leq C(Z_0 - Z_\infty)^2 e^{-2bt} + C\|u_0 - Z_\infty \mathcal{U}(\mu; \cdot)\|^2 e^{-2bt}.$$

Using the Cauchy-Schwarz inequality as in (49), we find that

$$(Z_0 - Z_\infty)^2 \leq C\|u_0 - Z_\infty \mathcal{U}(\mu; \cdot)\|^2,$$

so

$$(W - 1)^2 + (Z - Z_\infty)^2 \leq C\|u_0 - Z_\infty \mathcal{U}(\mu; \cdot)\|^2 e^{-2bt}.$$

This inequality ensures that the terms (i) and (ii) decrease to zero exponentially fast. It only remains to prove the same result for (iii), and for this we use the explicit formula (51). We obtain

$$\begin{aligned} \int_0^\infty (\mathcal{U}(W^{-k}\mu^{-k}x) - \mathcal{U}(\mu^{-k}x))^2 (\phi(x) + x^r) dx &= C \int \left( e^{-\frac{\beta}{\gamma\mu}(W^{-k}x)^\gamma} - e^{-\frac{\beta}{\gamma\mu}x^\gamma} \right)^2 (x + x^r) dx \\ &= C \int \left( e^{-\frac{2\beta}{\gamma\mu}(W^{-k}x)^\gamma} + e^{-\frac{2\beta}{\gamma\mu}x^\gamma} - 2e^{-\frac{\beta}{\gamma\mu}(1+W^{-1})x^\gamma} \right)^2 (x + x^r) dx \\ &= C \int e^{-\frac{2\beta}{\gamma\mu}y^\gamma} (W^k y + W^{rk} y^r) W^k dy + \int e^{-\frac{2\beta}{\gamma\mu}x^\gamma} (x + x^r) dx \\ &\quad - 2 \int e^{-\frac{2\beta}{\gamma\mu}z^\gamma} \left( \left( \frac{1+W^{-1}}{2} \right)^{-k} z + \left( \frac{1+W^{-1}}{2} \right)^{-rk} z^r \right) \left( \frac{1+W^{-1}}{2} \right)^{-k} dz \\ &= C\psi_1(W) \int e^{-\frac{2\beta}{\gamma\mu}x^\gamma} x dx + C\psi_r(W) \int e^{-\frac{2\beta}{\gamma\mu}x^\gamma} x^r dx \end{aligned}$$

where

$$\psi_r(W) := W^{(r+1)k} + 1 - 2 \left( \frac{1+W^{-1}}{2} \right)^{-(r+1)k}.$$

Thanks to a Taylor expansion, we find that, locally,

$$|\psi_r(W)| \leq C(W - 1)^2$$

and so

$$\int_0^\infty (\mathcal{U}(W^{-k}\mu^{-k}x) - \mathcal{U}(\mu^{-k}x))^2 (x + x^r) dx \leq C\|u_0 - Z_\infty \mathcal{U}(\mu; \cdot)\|^2 e^{-2bt}.$$

Finally there exists a constant  $C > 0$  such that, for  $\|u_0 - Z_\infty \mathcal{U}(\mu; \cdot)\|$  small enough so that  $(W, Z)$  stay in the neighbourhood  $\mathcal{V}_\eta(W_\infty, Z_\infty)$ , we have

$$\begin{aligned} \|u(t, \cdot) - Z_\infty \mathcal{U}(\mu; \cdot)\| &\leq \|u(t, \cdot) - Q\mathcal{U}(W\mu; \cdot)\| + \|Q\mathcal{U}(W\mu; \cdot) - Z_\infty \mathcal{U}(\mu; \cdot)\| \\ &\leq C\|u_0 - Z_\infty \mathcal{U}(\mu; \cdot)\| e^{-bt} \end{aligned}$$

and it is the local exponential stability of the nontrivial steady states which satisfy  $f'(I_\infty) < 0$  in the case when  $\kappa \equiv 2$ .

When  $f'(I_\infty) > 0$ , the steady state  $(W_\infty, Z_\infty)$  is a saddle point of  $L$  so it is instable.  $\square$

We remark that the structure of the reduced system (39) is different for  $p < 1$  and  $p > 1$ . The nontrivial steady states are focuses in the case when  $p < 1$  and nodes for  $p \geq 1$  (see Figure 1 for a numerical illustration in the case of Corollary 1.3).

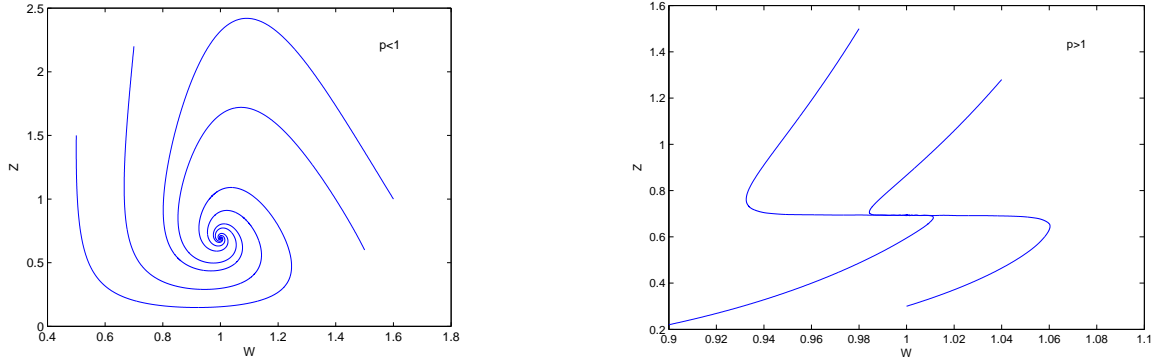


Figure 1: Solutions to System (39) are plotted in the phase plan  $(W, Z)$  for two different values of parameter  $p$ . The other coefficients are  $\gamma = 0.1$ ,  $\mu = 1$  and  $f(x) = 2e^{-x}$ . We can see that the steady state is a focus for  $p < 1$  (left) and a node for  $p > 1$  (right).

## 4 Nonlinear Drift and Death Terms: Stable Persistent Oscillations

We have seen in Theorem 1.1 that any solution to the nonlinear equation (8) converges to a steady state. Can this result be extended to Equation (10) where the death rate is also nonlinear? The result in Theorem 1.4 answer negatively to this question. Indeed it ensures the existence of functions  $f$  and  $g$ , and parameters  $p$  and  $q$  such that Equation (10) admits periodic solutions. Here we give examples of such functions and parameters and more precisely we prove, thanks to the Poincaré-Bendixon theorem, that any solution with an initial distribution in the eigenmanifold  $\mathcal{E}$  which is not a steady state converges to a periodic solution. Then we extend this result by surrounding this set of initial distributions by an open neighbourhood in  $\mathcal{H}$ .

In the proof, we need to know the dependency of some quantities on the parameters  $p$  and  $q$ . Since we do not know the dependencies of  $M_p = \int x^p \mathcal{U}(x) dx$  on  $p$ , we consider an equation slightly different from (10), namely

$$\frac{\partial}{\partial t} u(t, x) = -f \left( \frac{\int x^p u(t, x)}{\int x^p \mathcal{U}(x)} \right) \frac{\partial}{\partial x} (xu(t, x)) - g \left( \frac{\int x^q u(t, x)}{\int x^q \mathcal{U}(x)} \right) u(t, x) + \mathcal{F}_\gamma u(t, x). \quad (55)$$

Clearly the existence of functions  $f$  and  $g$  for which persistent oscillations appear in Equation (55) ensures the same result for Equation (10) (up to a dilation of  $f$  and  $g$ ). Now let set the assumptions on the two function  $f$  and  $g$  which allow to obtain periodic oscillations. We consider differentiable increasing functions  $f$  and  $g$  such that

$$f(0) > g(0) = 0 \quad \text{and} \quad f(\infty) < g(\infty) = \infty. \quad (56)$$

Moreover assume that  $g$  is invertible and define on  $\mathbb{R}^+$  the function

$$\psi(W) := f\left(W^{k(p-q)}g^{-1}(W)\right). \quad (57)$$

To ensure the existence and uniqueness of a nontrivial equilibrium, we assume that

$$\exists! W_\infty \geq 0, \quad \psi(W_\infty) = W_\infty \quad \text{and moreover} \quad \psi'(W_\infty) < 1. \quad (58)$$

This steady state is unstable if, denoting  $Q_\infty := W_\infty^{-kq}g^{-1}(W_\infty)$ , we have

$$Q_\infty \left( pW_\infty^{kp}f'(W_\infty^{kp}Q_\infty) - W_\infty^{kq}g'(W_\infty^{kq}Q_\infty) \right) - \frac{W_\infty}{k} > 0. \quad (59)$$

Then solutions to Equation (55) with initial distribution close to the set  $\mathcal{E} \setminus \{Q_\infty \mathcal{U}(W_\infty; \cdot)\}$  exhibit asymptotically periodic behaviors. More precisely, we have the following result

**Theorem 4.1.** *Consider increasing differentiable functions  $f$  and  $g$  satisfying conditions (56), (58) and (59), a parameter  $\gamma \in (0, 2]$  and a fragmentation kernel  $\kappa$  which satisfies Assumption (32). Then there exists an open neighbourhood  $\mathcal{V}$  of  $\mathcal{E} \setminus \{Q_\infty \mathcal{U}(W_\infty; \cdot)\}$  in  $\mathcal{H}$  such that, for any initial distribution  $u_0 \in \mathcal{V}$ , there exists periodic functions  $W(t)$  and  $Q(t)$  such that*

$$\|u(t, \cdot) - Q(t)\mathcal{U}(W(t); \cdot)\|_{\mathcal{H}} \xrightarrow[t \rightarrow \infty]{} 0. \quad (60)$$

Before proving this theorem, we give examples of functions  $f$  and  $g$  such that conditions (56), (58) and (59) are satisfied. These examples, together with Theorem 4.1, give the proof of Theorem 1.4.

**Example 1.** Assume that there exists  $C > 0$  such that for all  $x \geq 0$ ,  $g(x) \leq C x g'(x)$ , then  $\psi(W) = W$  has a unique solution for  $k(q - p) > C$ . Indeed, if we compute the derivative of  $\psi$  we find

$$\psi'(W) = W^{k(p-q)} \left( k(p-q) \frac{g^{-1}(W)}{W} + (g^{-1})'(W) \right)$$

and  $g(x) \leq C x g'(x)$  implies that  $x(g^{-1})'(x) \leq C g^{-1}(x)$ . So if  $k(q - p) > C$ ,  $\psi$  decreases and Assumption (58) is fulfilled. If moreover  $f(1) = g(1) = 1$ , then the unique nontrivial equilibrium is given by  $W_\infty = 1$ . Then condition (59) is satisfied for  $p > \frac{g'(1) + \frac{1}{k}}{f'(1)}$ .

**Example 2.** Consider the case  $g(x) = x$  and  $p = q$ , and assume that  $f(x) - x$  has a unique root  $x_0$  and  $f'(x_0) - 1 < 0$ . Then  $\psi(W) = f(W)$  and Assumption (58) is satisfied. Moreover, condition (59) writes  $pf'(W_\infty) > 1 + \frac{1}{k}$  so it is satisfied for  $p$  large enough.

Now we give a lemma useful for the proof of Theorem 4.1.

**Lemma 4.2.** *Consider a dynamical system in  $\mathbb{R}^n$  with a parameter  $\varepsilon(t)$*

$$\dot{X} = F(X; \varepsilon(t)) \quad (61)$$

*with  $F \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^n)$ . Assume that for any vanishing parameter  $\|\varepsilon(t)\| \xrightarrow[t \rightarrow \infty]{} 0$  the solutions to Equation (61) are bounded. Then for any solution  $X^\varepsilon$  associated to  $\|\varepsilon(t)\| \xrightarrow[t \rightarrow \infty]{} 0$ , there exists a solution  $X^0$  associated to  $\varepsilon \equiv 0$  such that  $X^\varepsilon$  and  $X^0$  have the same  $\omega$ -limit set.*

*Proof of Lemma 4.2.* Let  $X(t)$  a solution to System (61) with  $\|\varepsilon(t)\| \rightarrow 0$ . By assumption,  $X(t)$  is bounded, so  $\dot{X}(t)$  is also bounded since  $F$  is continuous. Now consider a sequence  $\{t_k\}_{k \in \mathbb{N}}$  which tends to the infinity and define the sequence  $\{W_k(\cdot)\}$  by  $W_k(t) = W(t + t_k)$ . This sequence is bounded in  $W^{1,\infty}(\mathbb{R}_+)$  so there exists a subsequence which converge to  $W_\infty(\cdot)$ . This limit is a solution to Equation (61) with  $\varepsilon \equiv 0$ . We take  $W^0 := W_\infty$  that ends the proof of Lemma 4.2.  $\square$

*Proof of Theorem 4.1.* We divide the proof in two parts: first the result for  $u_0 \in \mathcal{E}$  and then the existence of a neighbourhood  $\mathcal{V}$  of  $\mathcal{E}$  in  $\mathcal{H}$  where the result persists.

**First step:**  $u_0 \in \mathcal{E}$ .

For  $u_0 \in \mathcal{E} \setminus \{0\}$ , there are  $W_0 > 0$  and  $Q_0 > 0$  such that

$$u_0(x) = Q_0 \mathcal{U}(W_0; x).$$

Then, if  $u(t, x)$  is the solution to Equation (55) and  $W$  is the solution to

$$\dot{W} = \frac{W}{k} \left( f \left( \frac{M_p[u](t)}{M_p[\mathcal{U}]} \right) - W \right)$$

with  $W(0) = W_0$ , the relation holds for all  $t > 0$  and  $x > 0$

$$u(t, x) = Q(t) \mathcal{U}(W(t); x)$$

where  $Q(t) := Q_0 e^{\int_0^t (W(s) - g(M_q[u](s)/M_q[\mathcal{U}])) ds}$ . Then we can compute

$$M_p[u](t) = \int_0^\infty x^p u(t, x) dx = W^{kp}(t) Q(t) M_p[\mathcal{U}]$$

and finally we obtain the reduced system of ODEs satisfied by  $(W, Q)$

$$\begin{cases} \dot{W} &= \frac{W}{k} \left( f \left( W^{kp} Q \right) - W \right), \\ \dot{Q} &= Q \left( W - g \left( W^{kq} Q \right) \right). \end{cases} \quad (62)$$

We prove that System (62) has bounded solutions and a unique positive steady state which is unstable. Then we use the Poincaré-Bendixon theorem to ensure the convergence to a limit cycle.

The fact that  $0 < f(0) \leq f \leq f(\infty) < \infty$  and that  $g$  increases from 0 to the  $\infty$  ensures that the solution remains bounded. Let  $(W_\infty, Q_\infty)$  a positive steady state. It satisfies

$$W_\infty = f \left( W_\infty^{kp} Q_\infty \right) = g \left( W_\infty^{kq} Q_\infty \right)$$

and so, since  $g$  is invertible,  $Q_\infty = W_\infty^{-kq} g^{-1}(W_\infty)$ . Then  $W_\infty$  is solution to the equation

$$W_\infty = f \left( W_\infty^{k(p-q)} g^{-1}(W_\infty) \right) = \psi(W_\infty)$$

but Assumption (58) ensures the uniqueness of such a solution. Now look at the stability of this positive steady state. We write system (62) under the form

$$\begin{pmatrix} \dot{W} \\ \dot{Q} \end{pmatrix} = F \begin{pmatrix} W \\ Q \end{pmatrix},$$

and we have

$$Jac(F)_{eq} = \begin{pmatrix} pW_\infty^{kp}Q_\infty f'(W_\infty^{kp}Q_\infty) - \frac{1}{k}W_\infty & \frac{1}{k}W_\infty^{kp+1}f'(W_\infty^{kp}Q_\infty) \\ Q_\infty - kqW_\infty^{kq-1}Q_\infty^2 g'(W_\infty^{kq}Q_\infty) & -W_\infty^{kq}Q_\infty g'(W_\infty^{kq}Q_\infty) \end{pmatrix}.$$

The trace of this matrix is

$$T = Q_\infty \left( pW_\infty^{kp}f'(W_\infty^{kp}Q_\infty) - W_\infty^{kq}g'(W_\infty^{kq}Q_\infty) \right) - \frac{W_\infty}{k}$$

and the determinant is

$$D = \frac{W}{k}Q \left( W^{kq}g'(W^{kq}Q) - W^{kp}f'(W^{kp}Q) \right) + (q-p)W^{k(p+q)}Q^2 f'(W^{kp}Q)g'(W^{kq}Q).$$

We know from Assumption (58) that  $\psi'(W_\infty) < 1$  and, if we compute  $\psi'(W)$  we find

$$\psi'(W) = \left[ k(p-q)W^{k(p-q)-1}g^{-1}(W) + \frac{W^{k(p-q)}}{g'(g^{-1}(W))} \right] f'(W^{k(p-q)}g^{-1}(W)).$$

Since  $g^{-1}(W) = W^{kq}Q$  we finally obtain

$$D = \frac{1}{k}W_\infty^{kq+1}Q_\infty g'(W_\infty^{kq}Q_\infty)(1 - \psi'(W_\infty)) > 0.$$

Thus when  $T > 0$ , namely when Assumption (59) is satisfied, the two eigenvalues have positive real parts and the positive steady state is unstable. Now we prove that  $(W, Q)$  remains away from the boundaries of  $(\mathbb{R}_+)^2$ . For this we write that

$$\forall t > 0, \quad \underline{W} := \min(W_0, f(0)) \leq W(t) \leq \max(W_0, f(\infty)) := \overline{W}$$

and then

$$Q \geq \min(Q_0, \overline{W}^{-kp}g^{-1}(\underline{W})).$$

Since  $f(0) > 0$ ,  $\underline{W} > 0$  for  $W_0 > 0$  so any solution with  $W_0 > 0$  and  $Q_0 > 0$  stays at positive distance of the boundaries of  $(\mathbb{R}_+)^2$ . Then the Poincaré-Bendixon theorem (see [35] for instance) ensures any solution to System (62) with  $W_0 > 0$ ,  $Q_0 > 0$  and  $(W_0, Q_0) \neq (W_\infty, Q_\infty)$  converges to a limit cycle.

### Second step: Existence of $\mathcal{V}$ .

Let  $u_0 \not\equiv 0$  in  $\mathcal{H}$  and define from  $u(t, x)$  solution to Equation (55) with initial distribution  $u_0$  a function  $v$  by

$$v(h(t), x) = W^k(t)u(t, W^k(t)x)e^{\int_0^t (g(M_q[u](s)/M_q[\mathcal{U}]) - W(s)) ds}$$

with  $W$  solution to

$$\dot{W} = \frac{W}{k} \left( f \left( \frac{M_p[u]}{M_p[\mathcal{U}]} \right) - W \right)$$

and  $h$  solution to  $\dot{h} = W$  with  $h(0) = 0$ . We have already seen in Section 2.3 that  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is one to one since  $h(t) \geq k \ln(1 + \frac{t}{k})$ . We take  $W(0) = 1$  to have  $v(t = 0, \cdot) = u(t = 0, \cdot) = u_0$ . Thanks to Theorem 2.1 we know that  $v$  is solution to

$$\partial_t v(t, x) + \partial_x (x v(t, x)) + v(t, x) = \mathcal{F}_\gamma v(t, x)$$

and the GRE ensures the convergence

$$v(t, x) \xrightarrow[t \rightarrow \infty]{} \left( \int \phi(x) u_0(x) dx \right) \mathcal{U}(x).$$

As a consequence we have the equivalences, for any  $p \geq 0$ ,

$$M_p[u](t) \underset{t \rightarrow \infty}{\sim} \varrho_0 M_p[\mathcal{U}] W^{kp} e^{\int_0^t (W(s) - g(M_q[u](s)/M_q[\mathcal{U}])) ds}$$

so, if we define  $Q(t) := \varrho_0 e^{\int_0^t (W(s) - g(M_q[u](s)/M_q[\mathcal{U}])) ds}$ , we find that the reduced system (62) is “asymptotically equivalent” to Equation (55). More precisely we define as in the proof of Theorem 3.1

$$\varepsilon_p(t) = \frac{M_p[u](t)}{M_p[\mathcal{U}] W^{kp}(t) Q(t)} - 1$$

we have that  $\varepsilon_p \rightarrow 0$  and  $(W, Q)$  is solution to

$$\begin{cases} \dot{W} &= \frac{W}{k} \left( f \left( (1 + \varepsilon_p) W^{kp} Q \right) - W \right), \\ \dot{Q} &= Q \left( W - g \left( (1 + \varepsilon_q) W^{kq} Q \right) \right). \end{cases} \quad (63)$$

Now we prove that if  $W_0$  and  $Q_0$  are positive and  $(W_0, Q_0) \neq (W_\infty, Q_\infty)$ , then if  $\|u_0 - Q_0 \mathcal{U}(W_0; \cdot)\|_{\mathcal{H}}$  is small enough, the solution  $u$  to Equation (55) converges to a periodic solution. Denote by  $d$  the distance between  $(W_0, Q_0)$  and  $(W_\infty, Q_\infty)$ . Since  $(W_\infty, Q_\infty)$  is a source for System (62), there exists a ball with radius  $\rho < d$  such that the flux is outgoing, namely

$$\forall (W, Q) \in \partial B((W_\infty, Q_\infty), \rho), \quad F(W, Q) \cdot \mathbf{n} > 0 \quad (64)$$

where  $\mathbf{n}$  is the outgoing normal of  $B((W_\infty, Q_\infty), \rho)$ . Then if we define by  $F(W, Q; \varepsilon_p, \varepsilon_q)$  the flux of Equation (63), we have by continuity of  $f$  and  $g$  that there exists  $\varepsilon_0$  such that (64) remains true for  $F(W, Q; \varepsilon_p, \varepsilon_q)$  provided that  $\varepsilon_p$  and  $\varepsilon_q$  stay less than  $\varepsilon_0$ . But we know from the proof of Theorem 3.1 that there exists a constant  $C_p > 0$  such that for all time  $t > 0$ ,  $\varepsilon_p \leq C_p \|u_0 - Q_0 \mathcal{U}(W_0; \cdot)\|$ . So for  $\|u_0 - Q_0 \mathcal{U}(W_0; \cdot)\| \leq \frac{\varepsilon_0}{C_p + C_q}$ , the solution to System (63) cannot converge to the positive steady state  $(W_\infty, Q_\infty)$ . Thanks to the same arguments, if  $\|u_0 - Q_0 \mathcal{U}(W_0; \cdot)\|$  is small enough, then  $(W, Q)$  remains away from the boundaries of  $(\mathbb{R}_+)^2$ . We obtain thanks to Lemma 4.2 that for  $\|u_0 - Q_0 \mathcal{U}(W_0; \cdot)\|$  small enough,  $(W(t), Q(t))$  converges to a limit cycle  $(\tilde{W}(t), \tilde{Q}(t))$ . Then we write

$$\|u - \tilde{Q} \mathcal{U}(\tilde{W}; \cdot)\| \leq \|u - Q \mathcal{U}(W; \cdot)\| + \|Q \mathcal{U}(W; \cdot) - \tilde{Q} \mathcal{U}(\tilde{W}; \cdot)\|$$

and we conclude as in the proof of Theorem 3.1 that the solution  $u$  to Equation 10 converges in  $\mathcal{H}$  to  $\tilde{Q} \mathcal{U}(\tilde{W}; \cdot)$ . Finally we have proved, for any  $(W_0, Q_0) \in (\mathbb{R}_+^*)^2 \setminus \{(W_\infty, Q_\infty)\}$ , the existence of a ball centered in  $Q_0 \mathcal{U}(W_0; \cdot)$  such that any solution to Equation (55) with an initial distribution in this ball converges to a periodic solution. Then Theorem 4.1 is proved for  $\mathcal{V}$  the reunion of all these balls.  $\square$

To illustrate the convergence to a periodic solution for solutions to Equation (55), we plot in Figure 2 a solution to Equation (62) with an initial condition close to the steady state  $(W_\infty, Q_\infty)$  and for coefficients which satisfy the assumptions of Theorem 4.1.

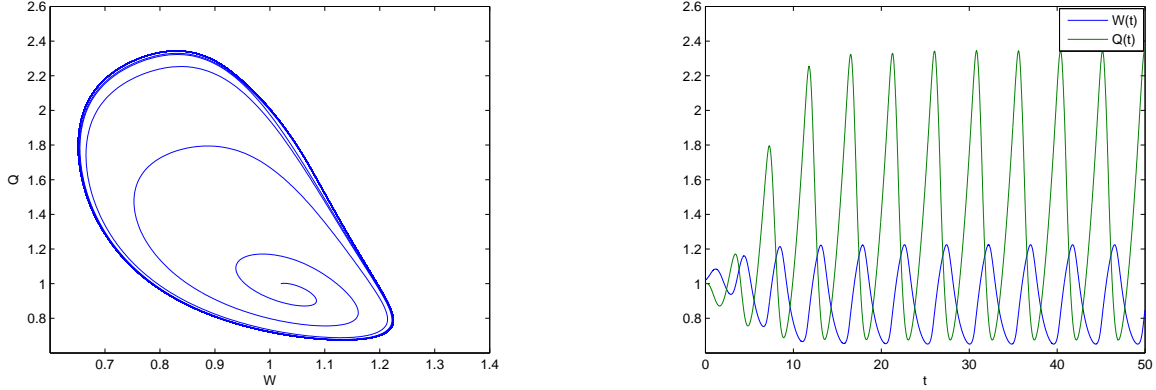


Figure 2: A solution to System (62) is plotted in the phase plan  $(W, Q)$  (left) and as a function of the time (right). The coefficients are  $\gamma = 1$ ,  $p = 2$ ,  $q = 5$ ,  $f(x) = 1 + e^{-1} - e^{-x^4}$  and  $g(x) = 0.9x$ .

## 5 The Prion Equation: Existence of Periodic Solutions

Prion diseases are believed to be due to self-replication of a pathogenic protein through a polymerization process not yet very well understood (see [40] for more details). To investigate the replication process of this protein, a mathematical PDE model was introduced by [32]. We recall this model here under a form slightly different from the original one (see [12, 20] for the motivations to consider this form)

$$\begin{cases} \frac{dV(t)}{dt} = \lambda - V(t) \left[ \delta + \int_0^\infty \tau(x)u(t, x) dx \right], \\ \frac{\partial}{\partial t}u(t, x) = -V(t) \frac{\partial}{\partial x}(\tau(x)u(t, x)) - \mu(x)u(t, x) + \mathcal{F}u(t, x). \end{cases} \quad (65)$$

In this equation,  $u(t, x)$  represents the quantity of polymers of pathogenic proteins of size  $x$  at time  $t$ , and  $V(t)$  the quantity of normal proteins (also called monomers). The polymers lengthen by attaching monomers with the rate  $\tau(x)$ , die with the rate  $\mu(x)$  and split into smaller polymers with respect to the fragmentation operator  $\mathcal{F}$ . The quantity of monomers is driven by an ODE with a death parameter  $\delta$  and production rate  $\lambda$ . This ODE is quadratically coupled to the growth-fragmentation equation because of the polymerization mechanism which is assumed to follow the mass action law.

This system admits a trivial steady state, also called disease-free equilibrium since it corresponds to a situation where no pathogenic polymers are present:  $V = \frac{\lambda}{\delta}$  and  $u \equiv 0$ . The stability of this steady state has been investigated under general assumptions on the coefficients in [11, 12, 56, 59]. The results indicate that the disease-free equilibrium is stable when it is the only steady state, and becomes unstable when a nontrivial steady state appears (also called endemic equilibrium). It is proved



in [9] that several such nontrivial steady states can exist. But the stability (even linear) of these steady states is a difficult and still open problem for general coefficients. The only existing results concern the “constant case” ( $\tau$  constant,  $\beta$  linear and  $\kappa$  constant) initially considered by [32], since then the model reduces to a closed system of ODEs. In this case, the problem has been entirely solved by [24, 32, 55]: the endemic equilibrium is unique and it is globally stable when it exists.

A new more general model has been introduced in [33] and takes into account the incidence of the *total mass* of polymers  $P(t) := \int x u(t, x) dx$  on the polymerization process. More precisely, they consider that the presence of many polymers reduces the attaching process of monomers to polymers by multiplying the polymerization rate by  $\frac{1}{1+\omega P(t)}$  with  $\omega$  a positive parameter. Then they prove similar results about the existence and stability of steady states, still in the case of constant parameters.

Here we look at a generalization of the influence of polymers on the polymerization rate by considering the system

$$\begin{cases} \frac{dV(t)}{dt} = -V(t)f\left(\int x^p u\right) \int_0^\infty \tau(x)u(t, x) dx - \delta V(t) + \lambda, \\ \frac{\partial}{\partial t}u(t, x) = -V(t)f\left(\int x^p u\right) \frac{\partial}{\partial x}(\tau(x)u(t, x)) - \mu u(t, x) + \mathcal{F}u(t, x), \end{cases} \quad (66)$$

where  $p \geq 0$  and  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a differentiable function. In this framework, the model of [33] corresponds to  $p = 1$  and  $f(P) = \frac{1}{1+\omega P}$ , together with  $\tau$  constant,  $\beta$  linear and  $\kappa$  constant. Using the reduction method to ODEs, we prove that such a system can exhibit periodic solutions. For this we consider the following system, where  $M_p = M_p[\mathcal{U}]$ ,

$$\begin{cases} \frac{dV(t)}{dt} = -V(t)f\left(\mu^{-kp}\frac{M_1}{M_p} \int x^p u\right) \int_0^\infty x u(t, x) dx - \delta V(t) + \lambda, \\ \frac{\partial}{\partial t}u(t, x) = -V(t)f\left(\mu^{-kp}\frac{M_1}{M_p} \int x^p u\right) \frac{\partial}{\partial x}(x u(t, x)) - \mu u(t, x) + \mathcal{F}_\gamma u(t, x), \end{cases} \quad (67)$$

which is a particular case of System (66) with coefficients satisfying the assumptions of Theorem 2.1 and up to a dilation of  $f$ . We prove that, under some conditions on the incidence function  $f$ , there exist values of  $p$  for which System (67) admits periodic solutions. The method is to consider  $p$  as a varying bifurcation parameter and prove that Hopf bifurcation occurs when  $p$  increases.

**Theorem 5.1.** *Define on  $[0, \frac{\lambda}{\mu})$  the function*

$$g(x) := \frac{\delta \mu}{\lambda - \mu^{k+1}x}$$

*and consider a positive differentiable function  $f$  satisfying*

$$\exists! x_0 > 0 \quad \text{s.t.} \quad f(x_0) = g(x_0) \quad \text{and moreover} \quad 0 < f'(x_0) < g'(x_0). \quad (68)$$

*Assume that  $\mu \leq (k + \mu^{-1})\delta$ , then there exists  $p > 0$  for which Equation (67) admits a periodic solution. More precisely, there exist  $V, W$  and  $Q$  periodic such that*

$$(V(t), Q(t)\mathcal{U}(W(t); x)) \text{ is solution to Equation (67).}$$

*Proof. First step: reduced dynamic in  $\mathcal{E}$*

We look at the dynamic of System (67) on the invariant eigenmanifold  $\mathcal{E}$ . For any initial condition  $u_0$  in  $\mathcal{E}$ , there exist  $Q_0$  and  $W_0$  such that  $u_0$  writes

$$u_0(x) = \frac{1}{M_1} Q_0 \mathcal{U}(W_0; x).$$

Consider the solution to System (67) corresponding to this initial data and define  $W$  as the solution to

$$\begin{cases} \dot{W} &= \frac{W}{k} \left( f \left( \mu^{-kp} \frac{M_1}{M_p} \int x^p u \right) V - W \right), \\ \dot{W}(0) &= W_0. \end{cases} \quad (69)$$

Then we know from Theorem 2.1 that the solution satisfies

$$u(t, x) = \frac{Q_0}{M_1} \mathcal{U}(W(t); x) e^{\int_0^t (W(s) - \mu) ds}$$

and this allows to compute

$$\int_0^\infty x^p u(t, x) dx = Q_0 \frac{M_p}{M_1} W^{kp} e^{\int_0^t (W(s) - \mu) ds}$$

and

$$\int_0^\infty x u(t, x) dx = Q_0 W^k e^{\int_0^t (W(s) - \mu) ds}.$$

Thus, if define  $Q(t) := Q_0 e^{\int_0^t (W(s) - \mu) ds}$ , Equation (69) becomes

$$\dot{W} = \frac{W}{k} \left( f \left( (\mu^{-1} W)^{kp} Q \right) V - W \right)$$

and System (67) reduces to

$$\begin{cases} \dot{V} &= \lambda - V \left( \delta + f \left( (\mu^{-1} W)^{kp} Q \right) W^k Q \right), \\ \dot{W} &= \frac{W}{k} \left( f \left( (\mu^{-1} W)^{kp} Q \right) V - W \right), \\ \dot{Q} &= Q (W - \mu). \end{cases} \quad (70)$$

Now we prove that System (70) admits a unique nontrivial steady state which undergoes a supercritical Hopf bifurcation when  $p$  increases from 0.

### Second step: Hopf bifurcation for the reduced system

First we look for a positive steady state of System (70). Such a steady state is unique and given by

$$\begin{aligned} W_\infty &= \mu \\ V_\infty &= \frac{1}{\delta} \left( \lambda - \mu^{k+1} Q_\infty \right) \end{aligned}$$

where  $Q_\infty$  satisfies

$$f(Q_\infty) = \frac{\delta \mu}{\lambda - \mu^{k+1} Q_\infty} =: g(Q_\infty).$$

Such a  $Q_\infty$  exists and is unique by Assumption (68), moreover it satisfies  $\lambda - \mu^{k+1}Q_\infty > 0$  and so  $V_\infty$  is positive. Finally there exists a unique positive steady state and we prove that a Hopf bifurcation occurs at this point. The linear stability of the steady state is given by the eigenvalues of the Jacobian matrix

$$Jac_{eq} = \begin{pmatrix} -\delta - \mu^k Q f(Q) & -k\mu^k V Q (f(Q) + p Q f'(Q)) & -\mu^k V (f(Q) + Q f'(Q)) \\ \frac{\mu}{k} f(Q) & p\mu V Q f'(Q) - \frac{\mu}{k} & \frac{\mu}{k} V f'(Q) \\ 0 & Q & 0 \end{pmatrix}$$

where the  $\infty$  indices are suppressed for the sake of clarity. The trace of this matrix is

$$T = -\delta - \mu^k Q f(Q) - \frac{\mu}{k} + p\mu V Q f'(Q)$$

which is negative for  $p < p_1$  and positive for  $p > p_1$  with

$$p_1 := \frac{\delta + \mu/k + \mu^k Q f(Q)}{\mu V Q f'(Q)} > 0.$$

The determinant is

$$D = \frac{\mu}{k} V Q \left( \delta f'(Q) - \mu^k f^2(Q) \right).$$

It is independent of  $p$  and negative since  $f'(x_0) < g'(x_0)$  and

$$g'(x_0) = \frac{\delta \mu^{k+2}}{(\lambda - \mu^{k+1} x_0)^2} = \frac{\mu^k}{\delta} f^2(x_0).$$

The sum of the three  $2 \times 2$  principal minors is

$$M = -\delta p V Q f'(Q) + \frac{\mu \delta}{k} + \mu^k \left( \mu + \frac{1}{k} \right) Q f(Q) - \frac{\mu}{k} V Q f'(Q).$$

To use the Routh-Hurwitz criterion, let define  $\psi(p) := MT - D$  and look at its sign. For  $p = 0$  we have

$$\begin{aligned} \psi(0) = -\frac{\mu}{k} \left[ \delta^2 + \delta Q f(Q) + \frac{\mu \delta}{k} + (\delta(k + \mu^{-1}) - \mu) \mu^k Q f(Q) \right. \\ \left. + V Q \left( \mu^{k-1} (k + \mu^{-1}) f^2(Q) - f'(Q) \right) \left( \mu^k Q f(Q) + \frac{\mu}{k} \right) \right], \end{aligned}$$

and it is negative since  $\mu \leq (k + \mu^{-1})\delta$  and  $f'(Q) < g'(Q) = \frac{\mu^k}{\delta} f^2(Q) \leq \mu^{k-1} (k + \mu^{-1}) f^2(Q)$ . For  $p = p_1$ , it is positive because  $\psi(p_1) = -D > 0$ . Now we investigate the variations of  $\psi$  between 0 and  $p_1$ . The first derivative of  $T$ ,  $D$  and  $M$  are given by

$$T'(p) = \mu V Q f'(Q), \quad M'(p) = -\delta V Q f'(Q), \quad D'(p) = 0,$$

and the second derivatives are all null

$$T''(p) = M''(p) = D''(p) = 0.$$

So we have

$$\psi''(p) = 2M'(p)T'(p) < 0$$

and  $\psi$  is concave. Thus there exists a unique  $p_0 \in (0, p_1)$  such that  $\psi(p_0) = 0$ . Now we can use the Routh-Hurwitz criterion (see [35] for instance). For  $0 \leq p < p_0$  we have  $T < 0$ ,  $D < 0$  and  $MT < D$ , so the steady state is linearly stable with one real negative eigenvalue and two complex conjugate eigenvalues with a negative real part. For  $p_0 < p < p_1$  we have  $T < 0$ ,  $D < 0$  and  $MT > D$ , so the steady state is linearly unstable with one real negative eigenvalue and two complex conjugate eigenvalues with a positive real part. The two conjugate eigenvalues cross the imaginary axis when  $p = p_0$  so there is a Hopf bifurcation at this point. To prove that a periodic solution appears with this bifurcation, it remains to check that the complex eigenvalues cross the imaginary axis with a positive speed (see [29] for instance). Denote by  $a \pm ib$  the two conjugate eigenvalues and  $c < 0$  the real one. We have to prove that the derivative  $a'(p_0) > 0$ . For this we express  $\psi(p)$  in terms of  $a(p)$ ,  $b(p)$  and  $c(p)$ , and we use the concavity of  $\psi$ . We have for any  $p$

$$T = 2a + c, \quad D = c(a^2 + b^2), \quad M = a^2 + b^2 + 2ac,$$

so

$$\psi(p) = 2a(a^2 + b^2) + 4a^2c + 2ac^2.$$

Then, using that  $a(p_0) = 0$  by definition of  $p_0$ , we obtain

$$\psi'(p_0) = 2(b^2 + c^2)a'.$$

But  $\psi'(p_0) > 0$  because  $\psi$  is concave and increasing on a neighbourhood of  $p_0$ , so necessarily  $a'(p_0) > 0$ . This proves the existence of a periodic solution  $(V, W, Q)$  to System (70) for a parameter  $p \geq p_0$  close to  $p_0$ . Then the functions  $V(t)$  and  $Q(t)\mathcal{U}(W(t), x)$  are periodic and solution to System (67).  $\square$

The question to know if such a periodic solution is stable is difficult, even for the reduced dynamic (70). Nevertheless we give in Figure 3 evidences that it should be the case. This simulation is made with parameters and a function  $f$  satisfying Assumption (68), for a value of parameter  $p > p_0$ . It seems to indicate that the periodic solution persists for  $p$  away from  $p_0$ .

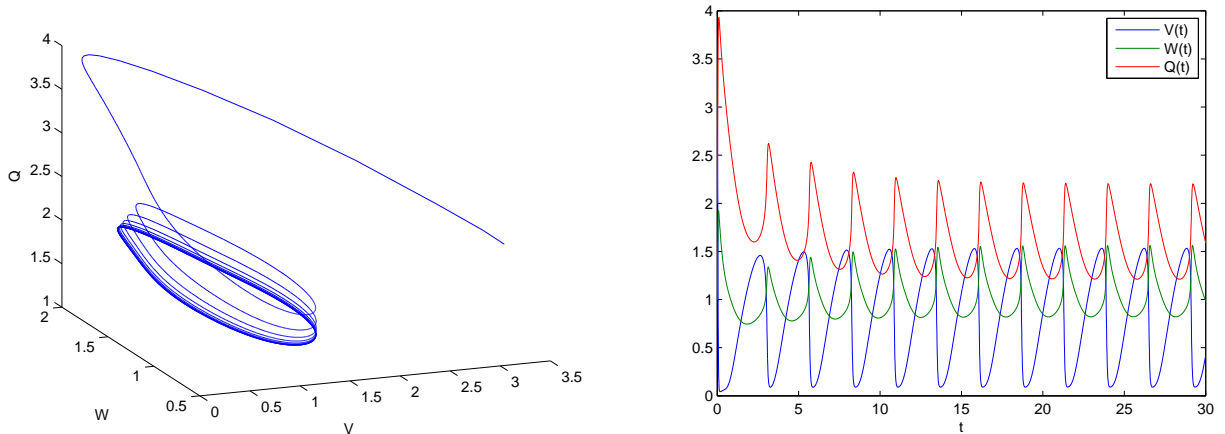


Figure 3: A solution to System (70) is plotted in the phase plan  $(W, Q)$  (left) and as a function of the time (right). The coefficients are  $\lambda = 0.9$ ,  $\delta = 0.2$ ,  $\mu = \gamma = 1$ ,  $f(x) = 6.3(1.1 - e^{-\frac{x^2}{20}})$  and  $p = 4$ .

## 6 Comparison between Perron and Floquet Eigenvalues

In this Section, we consider that the time-dependent terms  $V(t)$  and  $R(t)$  of the growth-fragmentation equation are  $T$ -periodic controls. Periodic controls are usually used in structured equations to model optimization problems. In the case of prion diseases (see Section 5), there exists an amplification protocole called PMCA (see [40] and references therein) which consists in periodically *sonicating* a sample of prion polymers in order to break them into smaller ones and thus increase their quantity. Between these phases of sonication, the sample is let in presence of a large quantity of monomers in order to allow a fast polymerization process. This protocole can be modeled by introducing in the growth-fragmentation equation a periodic control in front of the fragmentation operator [9, 10]. Then a problem is to find a periodic control which maximizes the proliferation rate of the polymers in the sample. Mathematically it can result into the problem to optimize the Floquet eigenvalue of the growth-fragmentation equation, namely the eigenvalue associated to periodic coefficients. Before solving this difficult question, a first step is to compare the Floquet eigenvalue to the Perron eigenvalue associated to constant coefficients, for instance the mean value on a period of the periodic ones, and to know if the Floquet one can be better than the Perron one. Such concerns are also investigated in the context of circadian rhythm for the optimization of chronotherapy (see [13, 14, 15]). The population is an age structured population of cells and the model is a system of renewal equations. The death and birth rates are assumed to be periodic, and the Floquet eigenvalue is compared to the Perron eigenvalue associated to geometrical or arithmetical time average of the periodic coefficients. Comparison results obtained show that the Floquet eigenvalue can be greater or less than the Perron one depending on parameters.

Here the controls are on the growth and death coefficients, and we give comparison results between Floquet and Perron eigenvalues in the case where  $\nu = 1$  or  $\nu = 0$  and  $\gamma = 1$ . The Floquet eigenelements  $(\Lambda_F, \mathcal{U}_F)$  associated to periodic controls are defined by two properties:  $\mathcal{U}_F(t, x)e^{\Lambda_F t}$  is a solution to Equation (12) and  $\mathcal{U}_F(t, x)$  is a  $T$ -periodic function of the time. For any  $T$ -periodic function  $f(t)$ , we use the notation

$$\overline{f} := \frac{1}{T} \int_0^T f(t) dt.$$

To ensure the uniqueness of Floquet eigenfunction, we impose  $\overline{\mathcal{U}_F} = 1$ . Then we have the following comparison results.

**Proposition 6.1.** *Assume that conditions (5)-(7) are satisfied with  $\nu = 1$ , and define, from  $V(t)$  the  $T$ -periodic control on growth,  $W(t)$  as the periodic solution to*

$$\dot{W} = \frac{\tau W}{k}(V - W). \quad (71)$$

*Then the identities hold*

$$\mathcal{U}_F(t, x) = \mathcal{U}(W(t); x) e^{\int_0^t (\Lambda(W(s), R(s)) - \Lambda_F) ds} \quad (72)$$

*and*

$$\Lambda_F = \Lambda(\overline{V}, \overline{R}) = \overline{\Lambda(V, R)}. \quad (73)$$

*Proof.* Thanks to Corollary 2.3, we have that

$$\mathcal{U}(W(t); x) e^{\int_0^t \Lambda(W(s), R(s)) ds} = \mathcal{U}_F(t, x) e^{\Lambda_F t}$$

and so, integrating in  $x$ , we find

$$e^{\Lambda_F t - \int_0^t \Lambda(W(s), R(s)) ds} = \int_0^\infty \mathcal{U}_F(t, x) dx.$$

Then, by periodicity, we have

$$e^{\Lambda_F T - \int_0^T \Lambda(W(s), R(s)) ds} = 1$$

which gives

$$\Lambda_F = \frac{1}{T} \int_0^T \Lambda(W(s), R(s)) ds = \frac{1}{T} \int_0^T (\tau W(s) - \mu R(s)) ds.$$

Thanks to ODE (71) we have

$$\int_0^T V - W = 0$$

and so

$$\Lambda_F = \frac{1}{T} \int_0^T (\tau V(s) - \mu R(s)) ds = \frac{1}{T} \int_0^T \Lambda(V(s), R(s)) ds.$$

□

In the case  $\nu = 0$  and  $\gamma = 1$ , we cannot ensure the existence of Floquet eigenelements with our method. Nevertheless, thanks to Corollary 2.5, we can compare the eigenvalues of the reduced system (26) which is satisfied by  $M_0[u]$  and  $M_1[u]$ .

**Proposition 6.2.** *Assume that  $\tau(x) = \tau$ ,  $\beta(x) = \beta x$ ,  $\mu(x) = \mu$  and  $\kappa$  is symmetric, then we have the comparison*

$$\overline{\Lambda(V, R)} \leq \Lambda_F \leq \Lambda(\overline{V}, \overline{R}). \quad (74)$$

*Proof.* Define  $W$  as the periodic solution to

$$\dot{W} = \frac{\Lambda(W, 0)}{k} (V - W).$$

We know thanks to Corollary 2.5 that

$$\mathcal{M}_0[W](t) = M_0[\mathcal{U}] e^{\int_0^t \Lambda(W(s), R(s)) ds}$$

and

$$\mathcal{M}_1[W](t) = M_1[\mathcal{U}] W^k(t) e^{\int_0^t \Lambda(W(s), R(s)) ds}$$

are solution to System (26). As a consequence

$$\Lambda_F = \frac{1}{T} \int_0^T \Lambda(W(s), R(s)) ds = \frac{1}{T} \int_0^T (\sqrt{\beta \tau W(s)} - \mu R(s)) ds.$$

Using the ODE satisfied by  $W$ , we have

$$0 = \int_0^T \frac{\dot{W}}{W} = \frac{\sqrt{\beta \tau}}{k} \int_0^T \frac{V - W}{\sqrt{W}}$$

and we obtain that

$$\int_0^T \sqrt{W} = \int_0^T \frac{V}{\sqrt{W}}.$$

Then the Cauchy-Schwartz inequality gives

$$\int_0^T \sqrt{V} = \int_0^T \frac{\sqrt{V}}{W^{\frac{1}{4}}} W^{\frac{1}{4}} \leq \sqrt{\int_0^T \frac{V}{\sqrt{W}}} \sqrt{\int_0^T \sqrt{W}} = \int_0^T \sqrt{W}$$

and so

$$\frac{1}{T} \int_0^T \Lambda(V, R) = \frac{1}{T} \int_0^T \sqrt{\beta\tau V} - \mu R \leq \frac{1}{T} \int_0^T \sqrt{\beta\tau W} - \mu R = \frac{1}{T} \int_0^T \Lambda(W, R) = \Lambda_F.$$

To obtain the second inequality in (74), we write using the ODE satisfied by  $W$

$$0 = \int_0^T \frac{\dot{W}}{\sqrt{W}} = \frac{\sqrt{\beta\tau}}{k} \int_0^T V - W$$

and so

$$\int_0^T V = \int_0^T W.$$

Thus we have, using the Jensen inequality,

$$\frac{1}{T} \int_0^T \sqrt{W(s)} ds \leq \sqrt{\frac{1}{T} \int_0^T W(s) ds} = \sqrt{\frac{1}{T} \int_0^T V(s) ds}$$

and finally

$$\Lambda_F = \frac{1}{T} \int_0^T (\sqrt{\beta\tau W(s)} - \mu R(s)) ds \leq \sqrt{\frac{1}{T} \int_0^T \beta\tau V(s) ds} - \frac{1}{T} \int_0^T \mu R(s) ds = \Lambda(\bar{V}, \bar{R}).$$

□

## Conclusion and Perspectives

We have introduced a new reduction method to investigate the long-time asymptotic behavior for growth-fragmentation equations with a nonlinear growth term. It allows to prove convergence and stability results but also to prove the existence of periodic solutions by using ODE methods. The technique is based on self-similar dependencies which require powerlaw coefficients, and the main theorem requires moreover a linear size-dependency of the growth coefficient. Nevertheless, we have given some results about the case when the growth rate is only assumed to be a powerlaw. A step further would be to adapt the ideas in this paper to build nonlinear entropy techniques adapted to general powerlaw growth coefficients.

## Acknowledgment

The author would like to thank Jean-Pierre Franoise for helpful discussions about limit cycles.

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